# §3 The homotopy extension property (HEP)

**Def. 3.1** A map $i: A \longrightarrow B$ has the <u>HEP</u> for a space $Z$ if for each commutative diagram of solid arrows

$$f \circ i = H \circ i_0$$

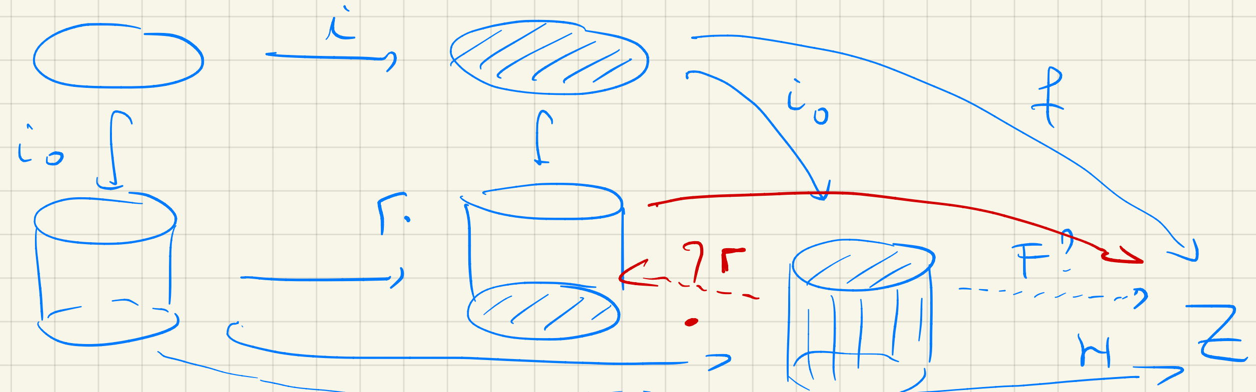
$$\begin{tikzcd}
a \arrow[d, mapsto] & A \arrow[r, "i"] \arrow[d, "i_0"] & B \arrow[d, "i_0"] \arrow[ddr, bend left, "f"] \arrow[dr, phantom, "1"] & \\
(a,0) & A \times I \arrow[r, "i \times I"] \arrow[drr, bend right, "H"'] \arrow[dr, phantom, "2"] & B \times I \arrow[dr, dashed, "F"] & \\
& & & Z
\end{tikzcd}$$

there exists a (dotted) map $F: B \times I \longrightarrow Z$ extending the homotopy $H$ (② commutes) and starting at $f$ (① commutes)

**Def. 3.2** A map $i$ is a <u>cofibration</u> if it has the HEP for all spaces $Z$

**Example 3.3** The inclusion $i: S^{n-1} \subset D^n$ is a cofibration

$n=2$:



Instead of solving the HE problem for all spaces $Z$ we do it for $\mathrm{Cyl}(i) = S^{n-1} \times I \amalg D^n / (x,0) \sim i(x)$

$$\begin{tikzcd}
S^{n-1} \arrow[r, "i"] \arrow[d, "i_0"'] & D^n \arrow[d, "i_0"] \arrow[ddr, "y"] & \\
S^{n-1} \times I \arrow[r, "i \times I"] \arrow[drr, bend right] & D^n \times I \arrow[dr, "r", red] & \\
& & \mathrm{Cyl}(i)
\end{tikzcd}$$

$y \mapsto \bar{y} \in D^n$

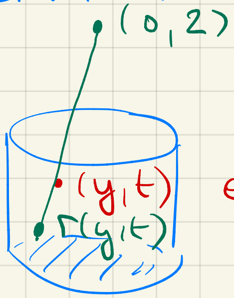
$(x,t) \longmapsto \overline{(x,t)}$

We construct a retraction $r$ of the inclusion $\mathrm{Cyl}(i) \subset D^n \times I$ which solves this HE problem and can then be used by composition to solve all HE problems.

The retraction $r$ is constructed as follows:

$\underline{n=2:}$

$\mathrm{Cyl}(i)$




$\in D^n \times I$

For any $(y,t) \in D^n \times I$, $r(y,t)$ is the intersection point of the line through $(y,t)$ and $(0,2)$ and $\mathrm{Cyl}(i)$.

By construction $r|_{\mathrm{Cyl}(i)} = \mathrm{id}$, thus the diagram commutes.

**Lemma 3.4** (Recognition Principle)
A map $i$ is a cofibration iff it has the HEP for $\mathrm{Cyl}(i)$.

proof: $(\Longrightarrow)$ obvious

$(\Longleftarrow)$

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & B \arrow[d] \arrow[ddr] \arrow[dddrr, bend left] \arrow[ddddrrr, bend left, "f"] & & & \\
A\times I \arrow[r] \arrow[drr] \arrow[ddrrr, bend right] \arrow[dddrrrr, bend right, "H"'] & Cyl(i) \arrow[dr, dashed, "j"] & & & \\
& & B\times I \arrow[dr, "r"] & & \\
& & & Cyl(i) \arrow[dr, dashed, "G"] & \\
& & & & Z
\end{tikzcd}$$

$j$ exists by the universal property for a pushout. We assume that it has a retraction $r$. Now given $f$ and $H$, we need to find an extension $F: B\times I \longrightarrow Z$. By universal property of the pushout we have $G: Cyl(i) \longrightarrow Z$ and we set $F = G\circ r$. $\square$

<u>Note</u>: Any cofibration is an embedding

## <u>Proposition 3.5</u>

The pushout of a cofibration along any map is a cofibration

<u>proof</u>: Let $i: A \hookrightarrow B$ be a cofibration, $f: A \longrightarrow X$ any map. Form the pushout

$$\begin{tikzcd}
A \arrow[r, hook, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
X \arrow[r, "j"'] & Y
\end{tikzcd}$$

, we claim $j$ is a cofibration.

Construct a commutative cube and solve the HE problem for
$\varphi : Y \longrightarrow Z$ and
$\Psi : X \times I \longrightarrow Z$

$$\begin{tikzcd}
A \arrow[rr, hook, "i"] \arrow[dr, "f"] \arrow[dd, "i_0"'] & & B \arrow[dr, "g"] \arrow[dd, "i_0"] \arrow[dddrr, bend left, "\beta"] & & \\
& X \arrow[rr, "j"] \arrow[dd, "i_0"] & & Y \arrow[dd, "i_0"] \arrow[ddr, bend left, "\varphi"] & \\
A \times I \arrow[rr, "i \times I"] \arrow[dr, "f \times I"'] \arrow[drrrr, bend right=40, "H"'] & & B \times I \arrow[dr, "g \times I"] \arrow[drr, "F"] & & \\
& X \times I \arrow[rr, "j \times I"'] \arrow[rrr, bend right, "\Psi"'] & & Y \times I \arrow[r, dashed, "\Phi"] & Z
\end{tikzcd}$$

The top face is a pushout by assumption, the bottom face is so as well because $- \times I$ is a left adjoint (mapping into $C$ corresponds to mapping from the top face into $\mathrm{map}(I, C)$ )

Let $\beta = \varphi \circ g$ and $H : \Psi \circ (f \times I)$. Solve the HE problem the back face: obtain a homotopy $F : B \times I \longrightarrow Z$ making the whole diagram commute.
Let us check $F \circ (i \times I) = \Psi \circ (f \times I)$, the latter composite is $H$ by definition, so OK. We get thus $\Phi : Y \times I \longrightarrow Z$. We are done since $\Phi \circ i_0 = \varphi$ because $Y$ is a pushout and both

verify
$$\begin{tikzcd}
A \arrow[r, hook, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[rdd, bend left, "\beta"] & \\
X \arrow[r, no head] \arrow[rrd, bend right] & Y \arrow[rd, "\varphi"] \arrow[rd, shift right, "\Phi \circ i_0"'] & \\
& & Z
\end{tikzcd}$$
that this diagram commutes, hence $\Phi \circ i_0 = \varphi$.

Finally, $\Phi \circ (f \times I) = \Psi$ by design. $\square$

$$\begin{tikzcd}
A \arrow[rr, hook, "i"] \arrow[dr, "f"] \arrow[dd, "i_0"'] & & B \arrow[dr, "g"] \arrow[dd, "i_0"] \arrow[rrddd, bend left, "\beta"] & & \\
& X \arrow[rr, "j"] \arrow[dd, "i_0"] & & Y \arrow[dd, "i_0"] \arrow[rdd, "\varphi"] & \\
A \times I \arrow[rr, "i \times I"] \arrow[dr, "f \times I"] \arrow[rrrrd, bend right=60, "H"'] & & B \times I \arrow[dr, "g \times I"] \arrow[rrd, "F"] & & \\
& X \times I \arrow[rr, "j \times I"] \arrow[rrr, bend right, "\Psi"'] & & Y \times I \arrow[r, dashed, "\Phi"] & Z
\end{tikzcd}$$

## Lemma 3.6

Composition of cofibrations is a cofibration

<u>proof</u>: For a finite number of cofibrations, we only need to look at two

$$\begin{tikzcd}
A \arrow[r, hook, "i"] \arrow[d] & B \arrow[r, hook, "j"] \arrow[d] & C \arrow[d] \arrow[ddr, bend left, "f"] & \\
A\times I \arrow[r, hook] \arrow[drrr, bend right, "H"'] & B\times I \arrow[r, hook] \arrow[drr, dashed, "F"] & C\times I \arrow[dr, dashed, "G"] & \\
 & & & Z
\end{tikzcd}$$

$i, j$ cofibrations.

First find $F: B\times I$ since $i$ is a cofibration, second get $G$ since $j$ is a cofibration.

For an infinite composition, $A_0 \overset{i_1}{\hookrightarrow} A_1 \overset{i_2}{\hookrightarrow} A_2 \overset{i_3}{\hookrightarrow} \cdots$

define $A = \bigcup_{n\geq 0} A_n$ and we show $i: A_0 \hookrightarrow A$ is a cofibration

The proof goes by induction, solving the HE problem from $H_n: A_n \times I \to Z$ to $H_{n+1}: A_{n+1}\times I \longrightarrow Z$ and define $H = \bigcup_{n\geq 1} H_{n+1}$

$\square$

## Proposition 3.7

If $(X, A)$ is a relative CW-complex, then the inclusion $i: A \subset X$ is a cofibration.

**proof:** From Example 3.3, $S^{n-1} \subset D^n$ is a cofibration, hence

$$\begin{tikzcd} S^{n-1} \arrow[r, hook] \arrow[d] & D^n \arrow[d] \\ * \arrow[r, hook] & S^n \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

we get $* \hookrightarrow S^n$ is a cofibration ($S^n$ is well-pointed). Also

$\coprod_\alpha S^{n-1}_\alpha \hookrightarrow \coprod_\alpha D^n_\alpha$ is a cofibration. Therefore the pushout

$$\begin{tikzcd} \coprod_\alpha S^{n-1}_\alpha \arrow[r, hook] \arrow[d] & \coprod_\alpha D^n_\alpha \arrow[d] \\ A \arrow[r, hook] & A \cup (\bigcup_\alpha e^n_\alpha) \end{tikzcd}$$

used to attach $n$-cells yields a cofibration $A \hookrightarrow A \cup (\bigcup_\alpha e^n_\alpha)$ by Proposition 3.5.

Iterating, we use Lemma 3.6 to conclude that $A \hookrightarrow X$ is a cofibration. $\square$

**Important cases:** $X_n \subset X$ is a cofibration

$X_n \subset X_{n+1}$ is a cofibration.

$A$ sub CW-cx, $A \subset X$ is a cofibration.

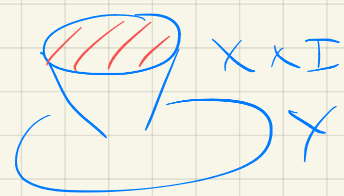
# How to turn a map into a cofibration?

Let $f: X \to Y$ and we wish to factorize $f$ as a composite

$$f: X \xhookrightarrow{i} Y' \xrightarrow[h]{\sim} Y$$

where $i$ is a cofibration and $h$ a homotopy equivalence

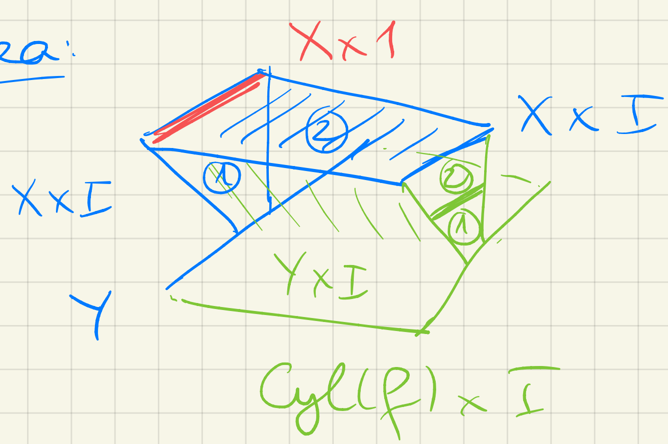
We take $Y' = Cyl(f)$ , $i: X \to Y'$
$x \mapsto \overline{(x,1)}$




$h: Y' \to Y$ collapses the cylinder
$\bar{y} \mapsto y$
$\overline{(x,t)} \mapsto f(x)$

**Lemma 3.8**
$i$ is a cofibration

idea:




We collapse $Y \times I$ onto $Y \times 0$
we send ① to $X \times I$ in $Cyl(f)$
② to $X \times I$ on top

"□"

# §4. Uniqueness of CW-approximation and Whitehead's Theorem

(Hatcher, 4.18, 4.19)

**Proposition 4.1**

Let $Z, Z'$ be two CW-approximations of $X$, via $f: Z \longrightarrow X$, $f': Z' \longrightarrow X$. There exists a map $h: Z \longrightarrow Z'$ s.t. $f' \circ h \simeq f$ in the pointed sense. Moreover this map $h$ is unique up to pointed homotopy.

proof: We turn $f'$ into a cofibration $Z' \xhookrightarrow{i} Cyl(f') \xrightarrow[\sim]{r} X$

$$Cyl(f') = X \amalg Z' \times I/\sim$$

Consider

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, dashed, "?"'] & X \arrow[d, "j"] \arrow[dr, bend left, "id_X"] & \\ Z' \arrow[r, hook, "i"] & Cyl(f') \arrow[r, "r", "\sim"'] & X \end{tikzcd}$$

We assume $X$ is path-connected, $x_0$ a chosen base point and by construction $z_0 = * = z_0'$. We can consider $j \circ f$ as a map of pairs $(Z, z_0) \longrightarrow (Cyl(f'), Z')$

The induction step: we have already changed $j \circ f$ to a map $h_n$ that sends $Z_n$ to $Z'$, and consider the $(n+1)$-cells.

$$\begin{tikzcd}
\bigvee_\alpha S^n_\alpha \arrow[r] \arrow[d, hook] & Z_n \arrow[r, "h_n|_{Z_n}"] \arrow[d] & Z' \arrow[d] \\
\bigvee_\alpha D^{n+1}_\alpha \arrow[r, "\Gamma"] & Z_{n+1} \arrow[r, "h_n|_{Z_{n+1}}"] & Cyl(f')
\end{tikzcd}$$

We get a family $(D^{n+1}_\alpha, S^n_\alpha) \longrightarrow (Cyl(f'), Z')$

But
$$\begin{tikzcd}
\pi_k Z' \arrow[r, "\cong"] \arrow[rr, bend right, "\cong \text{ CW-approx.}"'] & \pi_k\, Cyl(f') \arrow[r, "\cong \text{ h.eq.}"] & \pi_k X
\end{tikzcd}$$

By the les in $\pi_*$ we have $\pi_k(Cyl(f'), Z') = 0 \quad \forall k \geq 1$.
The Compression Lemma gives now a homotopic map to $(Z', Z')$. Hence we get also a homotopic map $Z_{n+1} \xrightarrow{h_{n+1}} Z'$

The HEP for $Z^{n+1} \hookrightarrow Z$ allows us to extend $h_{n+1}$ to $Z$. We end by induction.

Need still uniqueness!