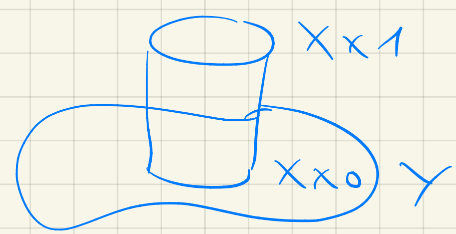
M. Postnikov (1927 - 2004), PhD Pontryagin (1949)

IV. Fibrations and Cofibrations

§1. Mapping cylinders and cones [tom Dieck, Chapter 4]

The mapping cylinder construction is useful to "turn any map into a cofibration".

Def. 1.1 Let $f: X \to Y$ be any map. The mapping cylinder
$$\mathrm{Cyl}(f) = (X \times I) \amalg Y \big/ (x,0) \sim f(x) \quad \forall x \in X.$$



This is the unpointed version, the pointed version replaces $X \times I$ by $X \rtimes I = X \times I / x_0 \times I$

This construction is functorial for morphisms of maps
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "\beta"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$
(i.e. pair $\alpha: X \to X'$, $\beta: Y \to Y'$ s.t. $\beta \circ f = f' \circ \alpha$)

Then α, β induce a map $\mathrm{Cyl}(f) \to \mathrm{Cyl}(f')$.

Lemma 1.2

The collapse map $r: Cyl(f) \longrightarrow Cyl(f)/(x,t)\sim(\tilde{x},0) \;\forall x$
is a homotopy equivalence

proof. The inclusion $i: Y \overset{i_2}{\hookrightarrow} (X\times I)\amalg Y \longrightarrow Cyl(f)$ has r as a retraction. Clearly $r\circ i = id_Y$ and $i\circ r: Cyl(f) \longrightarrow Cyl(f)$ is homotopic to the identity.

via
$$
\begin{aligned}
H: \underline{Cyl(f)\times I} &\longrightarrow \underline{Cyl(f)} \\
((x,s),t) &\longmapsto (x,st) \\
(\overline{y},t) &\longmapsto \overline{y}
\end{aligned}
$$

"shrink down the cylinder $X\times I$". $\square$

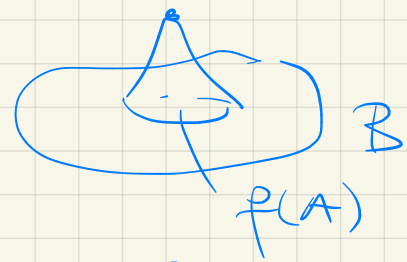
Def 1.3 The mapping cone $C(f)$ is $Cyl(f)/X\times 1$

Def. 1.4 A sequence of pointed maps $A \xrightarrow{f} B \xrightarrow{g} C$ is h-coexact if $[C,Z] \xrightarrow{g^*} [B,Z] \xrightarrow{f^*} [A,Z]$ is exact as sets ($\operatorname{Im} g^* = (f^*)^{-1}(c_{z_0})$) for any pointed space Z.

Lemma 1.5

The sequence $A \xrightarrow{f} B \xrightarrow{i} C(f)$ is h-coexact

<u>proof:</u> First $i \circ f \simeq *$, so $\mathrm{Im}\, i^* \subset (f^*)^{-1}([c])$



By construction we have a pushout

$$\begin{tikzcd}
a \arrow[d, mapsto] & A \arrow[r, "f"] \arrow[d, "i_0"'] & B \arrow[d, "i"] \arrow[ddr, bend left, "\beta"] & \\
(a,0) & CA \arrow[r] \arrow[drr, bend right, "H"'] & C(f) \arrow[dr, dashed, "\gamma"] & \\
 & & & Z
\end{tikzcd}$$

maps from $C(f)$ correspond to pairs $\beta : B \to Z$

$H : CA \to Z$

s.t. $\beta \circ f = H \circ i_0$

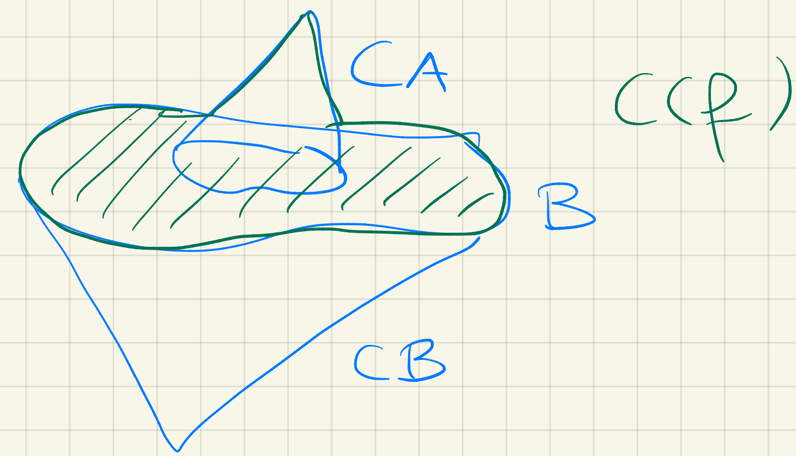
If γ is the induced map, $\gamma \circ i = \beta$, i.e. $i^*[\gamma] = [\beta]$.
Since H can be seen as a null-homotopy from $\beta \circ f$ to a constant map, we have shown that any $[\beta] \in (f^*)^{-1}[c]$ lies in $\mathrm{Im}\, i^*$. $\square$

Now we iterate this construction and construct the mapping cone of $i = f_1 : B \to C(f)$, then identify it with ΣA.

Lemma 1.6

$$C(f_1) \simeq \Sigma A$$

$C(f_1)$:



proof: Since

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "f"'] & CA \arrow[d] \\ B \arrow[r, hook] & C(f) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout diagram, and because $A \subset CA$, $B \subset C(f)$ are subspaces, the quotient spaces $CA/A \approx C(f)/B$ are homeomorphic. Now iterate

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "f"'] & CA \arrow[r] \arrow[d] & CA/A = \Sigma A \arrow[d] \\ B \arrow[r, hook, "f_1"] \arrow[d, hook] & C(f) \arrow[r, "p"] \arrow[d] \arrow[ul, phantom, "\ulcorner", very near start] & C(f)/B \approx \Sigma A \arrow[d] \\ CB \arrow[r, hook] & C(f_1) \arrow[r, "q"] \arrow[ul, phantom, "\ulcorner", very near start] & C(f_1)/CB \simeq \Sigma A \end{tikzcd}$$

We will use $C(f_1)/CB$ as a model for ΣA. We show that q is a homotopy equivalence. Define $s : \Sigma A \longrightarrow C(f_1)$

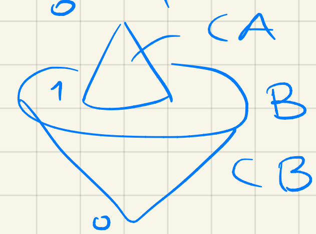
$$(a,s) \longmapsto \begin{cases} (a, 2s) & s \le \frac{1}{2} \\ (f(a), 2(1-s)) & s \ge \frac{1}{2} \end{cases}$$

Construct $H : C(f_1) \times I \longrightarrow C(f_1)$

$$(a, s, t) \longmapsto \begin{cases} (a, (1+t)s) & \text{if } (1+t)s \le 1 \\ (f(a), 2-(1+t)s) & \text{if not} \end{cases}$$

$$(b, s, t) \longmapsto (b, (1-t)s)$$

We check it's continuous: • if $(1+t)s = 1$, $(a, 1)_0 \sim (f(a), 1)$



• $(a, 1, t) \longmapsto (f(a), 1-t)$

$(f(a), 1, t) \longmapsto (f(a), 1-t)$ " ✓.

H is a homotopy from $H(\_, 0)$ to $H(\_, 1)$.

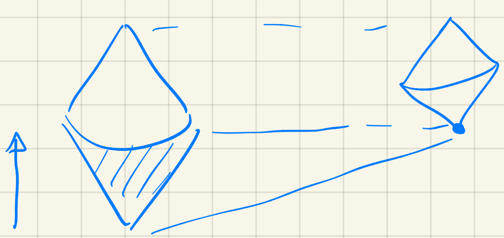
At $t = 0$, we get $H(a, s, 0) = (a, s)$, $H(\_, 0) = Id$.
$H(b, s, 0) = (b, s)$

At $t = 1$, we get $H(a, s, 1) = \begin{cases} (a, 2s) \\ (f(a), 2(1-s)) \end{cases}$, $H(b, s, 1) = (b, 0)$

So $H(\_, 1)$ is $s \circ q$. Hence $s \circ q \simeq id$.

We also have $q \circ s \simeq id$, $q \circ s : \overline{\Sigma A} \longrightarrow \overline{\Sigma A}$

$\overline{(a, s)} \longmapsto \overline{(a, \min(2s, 1))}$



$\square$

We continue with the cone of $f_2 : C(f) \to C(f_1)$.

Proposition 1.7
We have $C(f_2) \simeq \Sigma B$ and the induced map $f_3 : C(f_1) \to C(f_2)$ is homotopic to $\Sigma A \xrightarrow{\Sigma f \circ \imath} \Sigma B$ where $\imath$ is the inverse of the H-cogroup ΣA.

proof: The first claim follows from Lemma 1.6. Consider

$$\begin{tikzcd} B \arrow[r, "f_1"] & C(f) \arrow[r, "f_2"] \arrow[dr, "p"'] \arrow[r, phantom, shift right=4, "\circlearrowright"] & C(f_1) \arrow[r, "f_3"] \arrow[d, "q", "\simeq"'] \arrow[dr, phantom, "1"] & C(f_2) \arrow[d, "q'", "\simeq"'] \\ & & \Sigma A \arrow[r, "\Sigma f \circ \imath"] & \Sigma B \end{tikzcd}$$

Just like q collapses CB in $C(f_1)$,
q' ——— $C[C(f)]$ in $C(f_2)$. We show 1 commutes up to homotopy. Since $q' \circ f_3 = p' : C(f_2) \to \Sigma B$ collapses $C(f)$ in $C(f_2)$. To make it easier we precompose with s, a homotopy equivalence

$$\begin{tikzcd} & C(f_1) \arrow[r, "p'"] \arrow[d, "q", "\simeq"'] \arrow[dr, phantom, "1"] & \Sigma B \\ \Sigma A \arrow[ur, "s", "\simeq"'] \arrow[r, "id"'] \arrow[r, phantom, shift left=3, "h"] & \Sigma A \arrow[ur, "\Sigma f \circ \imath"'] & \end{tikzcd}$$

Let's compute $p' \circ s : \Sigma A \longrightarrow \Sigma B \cup C(f)$

$$\overline{(a,s)} \longmapsto \begin{cases} p'(\overline{a, 2s}) = * \\ p'(\overline{f(a), 2(1-s)}) = \overline{(f(a), 2(1-s))} \end{cases}$$

This is homotopic $\Sigma A \longrightarrow \Sigma B$

$$\overline{(a, s)} \longmapsto \overline{(f(a), 1-s)} = \Sigma f \circ \tau$$

where $\tau : \Sigma A \longrightarrow \Sigma A$.

$$(a, s) \longmapsto (a, 1-s)$$

$\square$

Theorem 1.8 (Puppe sequence)

The sequence $A \xrightarrow{f} B \xrightarrow{f_1} C(f) \xrightarrow{p} \Sigma A \xrightarrow{\Sigma f \circ \tau} \Sigma B \xrightarrow{\Sigma f_1 \circ \tau} \Sigma C(f)$

$\longrightarrow \Sigma^2 A \xrightarrow{\Sigma^2 f} \Sigma^2 B \longrightarrow \cdots$ is h-coexact at each space.

proof: This follows directly from Lemma 1.6 and Prop. 1.7 together with the fact $\tau \circ \tau = \mathrm{id}$. $\square$

Remark 1.9: On $[-, Z]$, $[\Sigma A, Z]$ and $[\Sigma B, Z]$ are groups, $(\Sigma f)^*$ and $(\Sigma f \circ \tau)^*$ have same image, same kernel. So we can forget τ if we prefer.

D. Puppe (1930 – 2005). Sequence is from 1958
PhD Seifert, Heidelberg

§2. Path spaces and loop spaces

Def 2.1 A sequence of pointed spaces $X \xrightarrow{f} Y \xrightarrow{g} Z$ is <u>h-exact</u> if $[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z]$ is exact for any pointed space A.

Def 2.2 For X a pointed space, $F(X) = \mathrm{map}_*(I, X)$ where $0 \in I$ is the chosen base point.

Just like A is "nicely" contained in CA, $F(X)$ "nicely" surjects onto X via $ev_1 : F(X) \longrightarrow X$, $\omega \longmapsto \omega(1)$, when X path-connected

Given a map $f : A \longrightarrow X$, an extension of f to the cone $A \subset CA$ corresponds to a null-homotopy $f \simeq *$. By adjunction this in turn yields a lift $A \longrightarrow F(X)$
$\mathrm{map}_*(A \wedge I, X) \cong \mathrm{map}_*(A, F(X))$

Def. 2.3. The mapping fiber $F(f)$ is the pullback of
$$X \xrightarrow{f} Y \xleftarrow{ev_1} F(Y)$$

Facts:
- $F(f) \xrightarrow{f_1} X \xrightarrow{f} Y$ is h-exact
- $F(f_1) \simeq \Omega Y$
- $F(f_2) \xrightarrow{f_3} F(f_1)$ is h.e. to $\Omega X \longrightarrow \Omega Y$, the map $\iota \circ \Omega f$ where ι is the inverse of ΩY.

Theorem 2.4 (dual Puppe sequence, fiber sequence)
The sequence
$$\cdots \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \to \Omega F(f) \to \Omega X \xrightarrow{\iota \circ \Omega f} \Omega Y \to F(f) \to$$
$$\to X \xrightarrow{f} Y$$
is h-exact

Example: $A = S^0$ we get a long exact sequence in π_*
$$\pi_n X \to \pi_n Y \to \pi_{n-1} F(f) \to \pi_{n-1} X \to \pi_{n-1} Y \to \cdots$$