§ 3 CW-approximation

We will prove that every space can be replaced, up to weak equivalence, by a CW-complex.

Def 3.1 An unpointed map $f: X \longrightarrow Y$ is a *weak homotopy equivalence* (weak equivalence, w.e.) if for any choice of base point $x \in X$ we have isomorphisms $f_*: \pi_n(X; x) \xrightarrow{\cong} \pi_n(Y; f(x))$ for any $n \geq 0$.

Remark 3.2: This means that $\pi_0 X$ and $\pi_0 Y$ are in 1-1 correspondance via f, and for any path-connected component $X_0 \subset X$, f induces isomorphisms $\pi_n X_0 \to \pi_n Y_0$ $\forall n \geq 1$ where Y_0 is the path-comp. of Y containing $f(X_0)$ for our favorite base point (since $\pi_n(X_0; x_0) \cong \pi_n(X_0; x_1)$ for any $x_0, x_1 \in X_0$).

A homotopy equivalence is a w.e.

Theorem 3.3 CW-Approximation

Any space X admits a <u>CW-approximation</u>, ie a weak equivalence $f: Z \longrightarrow X$ where Z is a CW-complex.

<u>proof:</u> We can assume X is path-connected since we can CW-approximate any component separately. The proof is by induction on skeleta of Z.

The CW-complex Z has a single 0-cell, $Z_0 = \{*\}$, and we define $f_0 : Z_0 \longrightarrow X$ by choosing our favorite base point in X. This map induces a bijection on π_0.

Next we choose generators $\alpha_i \in \pi_1 X$ and representatives $a_i : S^1 \longrightarrow X$, loops at the chosen basepoint. We construct $Y_1 = \bigvee_i S^1_i$. The base point is Z_0 and we define $g_1 : Y_1 \longrightarrow X$ by setting $g_1|_{S^1_i} = a_i$ since $\mathrm{map}_*(\bigvee_i S^1_i, X) \cong \prod_i \mathrm{map}_*(S^1_i, X)$.

Y_1 is path-connected so g_1 induces a bijection on π_0 and by construction $(g_1)_*: \pi_1 Y_1 \longrightarrow \pi_1 X$ is surjective since $S^1 \xhookrightarrow{j_i} \bigvee_i S^1_i \xrightarrow{g_1} X$ is a_i, so $(g_1)_*[j_i] = \alpha_i$.

We have $(g_1)_*: \pi_1 Y_1 \cong \ast_{i \in I} \pi_1 S^1_i \cong F(I) \twoheadrightarrow \pi_1 X$

Let $K = \ker (g_1)_*$ and choose generators $\beta_k \in K$ and representatives $b_k: S^1 \longrightarrow Y_1 = \bigvee_{i \in I} S^1_i$.

Construct the pushout

$$\begin{tikzcd}
\bigvee_k S^1_k \arrow[r, "\vee b_k"] \arrow[d, hook] & Y_1 \arrow[d, hook] \arrow[ddr, bend left, "g_1"] & \\
\bigvee_k D^2_k \arrow[r] \arrow[drr, dashed, "H"'] & Z_1 \arrow[ul, phantom, "\ulcorner"] \arrow[dr, dashed, "f_1"] & \\
& & X
\end{tikzcd}$$

We obtain $f_1: Z_1 \longrightarrow X$ by the universal property of the pushout. Consider an index k and the map

$$\begin{tikzcd}
S^1_k \arrow[r, "b_k"] \arrow[d, hook] & Y_1 \arrow[r, "g_1"] & X \\
D^2_k \arrow[urr, dashed, "R_k"'] \arrow[ur, phantom, "\circlearrowright"] & &
\end{tikzcd}$$

, which is null-homotopic since $(g_1)_*(\beta_k) = 1$. Hence there is an extension R_k. Set $H = \bigvee R_k$.

By construction the diagram with $\longrightarrow$ commutes and so $f_1 : Z_1 \longrightarrow X$ exists (and is unique).

Because $Y_1 \hookrightarrow Z_1 \xrightarrow{f_1} X$ is g_1, which is surjective on π_1, so is f_1. We use next the Seifert-van Kampen Theorem to understand that we have a p.o. of groups

$$\begin{tikzcd}
\pi_1(\bigvee_k S^1_k) = F(K) \arrow[r] \arrow[d] & \pi_1 Y_1 = F(I) \arrow[d] \\
\pi_1(\bigvee D^2_k) = 1 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \pi_1 Z_1 \cong F(I) \underset{F(K)}{*} 1 \cong F(I)/K
\end{tikzcd}$$

Hence the morphism $\pi_1 Z_1 \xrightarrow{\cong} \pi_1 X$ is an iso.

Let us assume that we have constructed a space Z_{n-1}, a CW-complex of dimension n and a map $f_{n-1} : Z_{n-1} \longrightarrow X$ s.t. $(f_{n-1})_* : \pi_k Z_{n-1} \xrightarrow{\cong} \pi_k X$ is an iso $\forall k \leq n-1$.

Let us choose generators $\alpha_i \in \pi_n X$ and representatives $a_i : S^n \longrightarrow X$. We define $Y_n = Z_{n-1} \vee (\bigvee_i S^n_i)$ and a map $g_n : Y_n \longrightarrow X$ by $g_n|_{Z_{n-1}} = f_{n-1}$ & $g|_{S^n_i} = a_i$

Just as before $(g_n)_* : \pi_n Y_n \longrightarrow \pi_n X$ is surjective since

$$\begin{tikzcd} S^n \arrow[r, "j_i"] \arrow[rrrr, bend right, "a_i"'] & \bigvee_i S^n_i \arrow[r, hook] & Z_{n-1} \vee (\bigvee_i S^n_i) = Y_n \arrow[rr, "g_n"] & & X \end{tikzcd}$$ is a_i.

To make it injective we kill the kernel $K = \ker((g_n)_*)$.

Choose generators $\beta_k \in K$ and representatives $b_k : S^n \longrightarrow Y_n$. Because

$$\begin{tikzcd} S^n \arrow[r, "b_k"] \arrow[d, hook] & Y_n \arrow[r, "g_n"] & X \\ D^{n+1} \arrow[urr, dashed, "h_k"'] & & \end{tikzcd}$$

is nullhomotopic ($(g_n)_*(\beta_k) = 0$)

there exists a null-homotopy h_k extending $g_n \circ b_k$ to D^{n+1}.

Define Z_n as pushout

$$\begin{tikzcd} \bigvee S^n_k \arrow[r, "\vee b_k"] \arrow[d, hook] & Y_n \arrow[d, hook] \\ \bigvee D^{n+1}_k \arrow[r] & Z_n = Y_n \cup \left(\bigcup_k e^{n+1}_k\right) \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

$$\begin{tikzcd} F(K) \arrow[r] \arrow[d] & F(I) \arrow[d] \arrow[ddr, bend left, "\mapsto g_i \in G"] & \\ 1 \arrow[r] \arrow[drr, bend right] & F(I)/\mathrm{Ker} \arrow[dr, dashed, "\exists!"] & \\ & & G \end{tikzcd}$$

$F(\gamma_k)$ $\gamma_k \longmapsto \beta_k$

Since Z_n has been constructed from Y_n by attaching $(n+1)$-cells, the pair (Z_n, Y_n) is n-connected, so the inclusion

$Y_n \hookrightarrow Z_n$ induces isos on $\pi_k Y_n \xrightarrow{\cong} \pi_k Z_n$ for $k \leq n-1$

Likewise (Y_n, Z_{n-1}) is $(n-1)$-connected, so

$\pi_k Z_{n-1} \xrightarrow{\cong} \pi_k Y_n$ for $k \leq n-2$.

Focus on π_{n-1}: The map $\pi_{n-1} Z_{n-1} \longrightarrow \pi_{n-1} Y_n$ is injective because $Y_n = Z_{n-1} \vee (\bigvee S^n_i)$ and Z_{n-1} is a retract of this wedge: $Z_{n-1} \hookrightarrow Y_n \longrightarrow Z_{n-1}$ is the identity.

collapse $\bigvee S^n_i$

It is also surjective: $\pi_{n-1}(Y_n, Z_{n-1}) = [(D^{n-1}, S^{n-2}), (Z_{n-1} \vee \bigvee S^n_i, Z_{n-1})]$

By cellular approximation, every map is homotopic to a map factoring through (Z_{n-1}, Z_{n-1}), hence zero.

All in all we have $Z_{n-1} \hookrightarrow Z_n$ induces iso on π_k, $k \leq n-1$

We finally need to show $\pi_n Z_n \xrightarrow{\cong} \pi_n X$ is an iso.

Again surjectivity is clear: $\pi_n Y_n \longrightarrow \pi_n Z_n \xrightarrow{(f_n)_*} \pi_n X$ is surjective by construction.

Let $x \in \ker(f_n)_* : \pi_n Z_n \longrightarrow \pi_n X$ and choose a representative
$k : S^n \longrightarrow Z_n = Y_n \cup \left(\bigcup e^{n+1}_k\right)$

By cellular approximation we might as well use a repr.
$k' : S^n \xrightarrow{k''} Y_n \hookrightarrow Z_n$

The map k'' represents a class $x'' \in \pi_n Y_n$ s.t.
$(g_n)_*(x'') = 0 \in \pi_n X$, ie $x'' \in \ker(g_n)_*$.

Therefore x'' is a sum of multiples of β_k, generators of the kernel: up to homotopy

$$\begin{tikzcd}
k'' \simeq S^n \arrow[r, "p"] \arrow[drrr, dashed, "k'"] & \bigvee S^n \arrow[r, "\vee b_{k_j}"] \arrow[d, hook] & \bigvee Y_n \arrow[r, "\nabla"] & Y_n \arrow[d, hook] \\
 & * \simeq \bigvee D^{n+1} \arrow[rr, "\vee k_{k_j}"'] & & Z_n
\end{tikzcd}$$

complete

We have seen that k' factors through a contractible space, hence $x = 0$. $\square$

<u>Remark 3.4</u> • There's a more general n-connected, relative (X, A). Here $A = *$, $n = 0$.
• Through the eyes of π_*, instead of Top_*, can use CW-cxs.

§4. Postnikov sections

Our aim is to construct for any space X a "tower"

$X \longrightarrow X[n]$ by killing all higher homotopy groups $\pi_k X$, $k \geq n+1$, without touching the lower ones.

<u>Def. 4.1</u> A space X s.t. $\pi_k X = 0$ for all $k \neq n$, and $\pi_n X \cong A$, for $n \geq 1$, is called an <u>Eilenberg-Mac Lane space</u> of type $K(A, n)$

<u>Example 4.2</u>
- S^1 is a $K(\mathbb{Z}, 1)$ $(\widetilde{S^1} = \mathbb{R} \simeq *)$
- $S^1 \times S^1$ is a $K(\mathbb{Z}\times\mathbb{Z}, 1)$
- $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}/_2, 1)$ $(\widetilde{\mathbb{RP}^\infty} = S^\infty \simeq *)$
- $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$ (no proof yet)

Eilenberg (1913-1998), PhD in Warsaw with Borsuk & Kuratowski, 1936.

Mac Lane (1909-2005), Yale.

Proposition 4.3 (Postnikov sections)
Let X be a path-connected space and $n \geq 1$. There exists a space $X[n]$ and a map $\ell_n : X \longrightarrow X[n]$ s.t. $\pi_k X[n] = 0$ for $k > n$ and $(\ell_n)_* : \pi_k X \xrightarrow{\cong} \pi_k X[n]$ is an iso for $k \leq n$.

proof: We choose generators $\alpha_i \in \pi_{n+1} X$ and repr. $a_i : S^{n+1} \longrightarrow X$. We construct a pushout

$$\begin{tikzcd}
\bigvee S^{n+1} \arrow[r, "\vee a_i"] \arrow[d, hook] & X \arrow[d, hook] \\
\bigvee D^{n+2} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X' = X \cup \left(\bigcup_i e^{n+2}\right)
\end{tikzcd}$$

The pair (X', X) is $(n+1)$-connected, hence $X \hookrightarrow X'$ induces an iso on π_k, $k \leq n$. Moreover $\pi_{n+1} X \to \pi_{n+1} X'$ is surjective. Any map $S^{n+1} \xrightarrow{g} X'$ factors through X by cellular approximation. As at the end of the CW-approx. we see that g is null-homotopic. So $\pi_{n+1} X' = 0$.
We conclude by induction. $\square$

Theorem 4.4 (Postnikov tower)

There is a tower of maps $X[n+1] \longrightarrow X[n]$ making the diagram commute

$$\begin{tikzcd}
 & X[n+1] \arrow[d] \\
X \arrow[ur, "\ell_{n+1}"] \arrow[r, "\ell_n"'] \arrow[ur, phantom, "\circlearrowleft", shift right=4] & X[n] \arrow[d] \\
 & \vdots \arrow[d] \\
 & X[1]
\end{tikzcd}$$

proof: $X[n+1]$ is constructed from X by attaching cells of dim $\geq n+3$. Let's look at one $(n+3)$-cell.

$$\begin{tikzcd}
S^{n+2} \arrow[r, "a"] \arrow[d, hook] & X \arrow[d, hook] \arrow[ddr, bend left, "\ell_n"] & \\
D^{n+3} \arrow[r] \arrow[drr, dashed, bend right, "H"'] & X' \arrow[dr, dashed, "\exists"] \arrow[ul, phantom, "\ulcorner"] & \\
 & & X[n]
\end{tikzcd}$$

$\ell_n \circ a \simeq *$ since $\pi_{n+2} X[n] = 0$

conclude inductively. $\square$