Def. 1.4 Let K be a polyhedron and $f : K \longrightarrow \mathbb{R}^k$. We say f is piecewise linear (PL) if there exists a decomposition of K into compact convex polyhedra $K_i \subset K$ s.t. $f|_{K_i}$ affine. A map $f : K \longrightarrow \mathring{e}^k$ is PL if it's so for some homeo $\mathring{e}^k \approx \mathbb{R}^k$.

Construction 1.5 (*) Let $f : I^n \longrightarrow \mathbb{R}^k$ be a map. Consider

$$B_1 = B(0,1) \subset B_2 = B(0,2) \subset \mathbb{R}^k.$$

Since I^n is compact and f is continuous, it is uniformly continuous. We choose $\varepsilon > 0$ s.t.

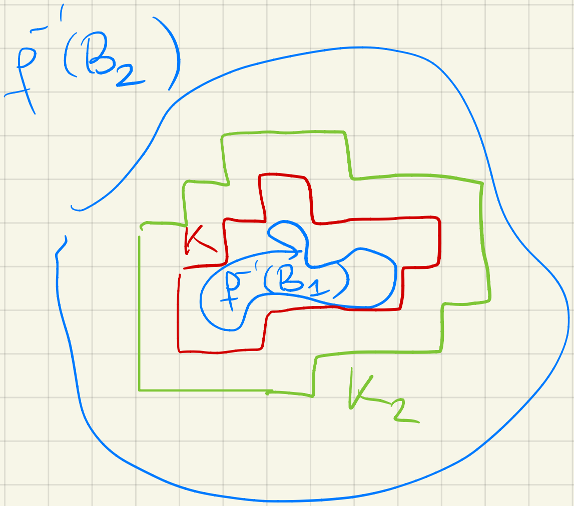
① $|x-y| < \varepsilon \Rightarrow |f(x) - f(y)| < 1/2$

② $\varepsilon < \frac{1}{2} d(f^{-1}(B_1), I^n \setminus f^{-1}(\mathring{B}_2))$

Choose $N \in \mathbb{N}$ s.t. $\operatorname{diam}[0, \frac{1}{N}]^n < \varepsilon$

(*) Assume B_2 is contained in the image of f.

Next subdivide I^n into small cubes of side $\frac{1}{N}$.



Set K = union of all small cubes which meet $f^{-1}(B_1)$

K_2 = ——————————— K

Every point of K_2 is at distance $< \varepsilon$ from K

so ——————— $< 2\varepsilon$ from $f^{-1}(B_1)$

Hence $K_2 \subset f^{-1}(B_2)$

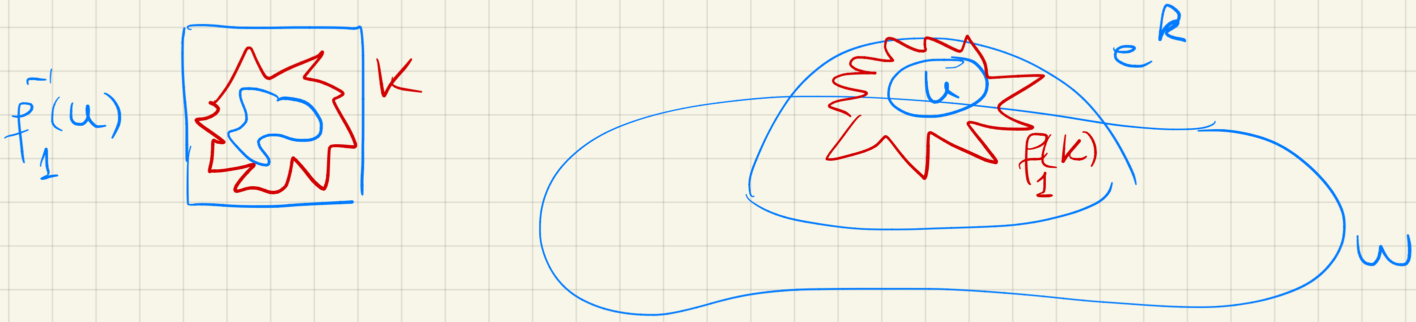
§ 2. Cellular approximation

Lemma 2.1

Let $Z = W \cup e^k$ and $f : I^n \longrightarrow Z$ a map (with $\mathring{e}^k \subset \operatorname{Im} f$). Then f is homotopic to a map f_1, relative to $f^{-1}(W)$ st. there exists a polyhedron $K \subset I^n$ and

(a) $f_1(K) \subset e^k$ and $f_1|_K$ is PL

(b) $\exists$ open $\emptyset \neq U \subset e^k$ st $f_1^{-1}(U) \subset K$.



<u>proof</u>: By Construction 1.5 for $B_1 \subset B_2 \subset \mathbb{R}^k \approx \mathring{e}^k$, we find polyhedra $K \subset K_2 \subset I^n$ ... Let us subdivide any small cube $\approx [0, \frac{1}{N}]^n$ into n-simplices Δ^n (barycentric subdivision).

Define a map g on K_2 by $g(v) = f(v)$ for any vertex v. We extend linearly to all simplices of K_2.

Define also a map $\varphi : K_2 \longrightarrow I$ with $\varphi|_K \equiv 1$ and $\varphi \equiv 0$ on the boundary of K and make it continuous for example as done for g.

Construct a homotopy $H : K_2 \times I \longrightarrow e^k$

$$H(\_, t) = (1 - t\varphi) \cdot f + t \cdot \varphi \cdot g$$

At $t = 0$: $H(\_, 0) = f$

$t = 1$: $H(\_, 1) = (1 - \varphi) \cdot f + \varphi g$, on K it's g.

$H(x,1) = g(x)$ if $x \in K$

$H(x,t) = f(x)$ if $x \in \partial K_2$, $\forall t \in I$

Thus we can extend H to a homotopy on I^n by using f constantly on $I^n \setminus K_2$. This defines a homotopy from f to a map f_1 which is g on K, a PL map.

First $f_1(K) \subset B_2 \subset e^k$ by construction, the rest of the proof consists in finding U. For this we prove that $0 \notin f_1(\overline{I^n \setminus K}) = C$, a compact subspace of Z, and then choose a (open) neighborhood U of 0 disjoint from C.

- On $I^n \setminus K_2$, $f_1 \equiv f$ sends a point outside of B_1
- Let σ be a simplex in $K_2 \setminus K_1$. By construction σ is contained in a small cube of radius $\frac{1}{N}$, $f(\sigma)$ is thus contained in a small ball of radius $\frac{1}{2}$.

To define g we used the same values as f on vertices and the image of cube is convex, so $g(\sigma)$ is contained in the same small ball. Likewise for $H(\sigma, t)$, for any t.

For $t=1$, we get $f_1(\sigma)$ is contained in this ball of radius $\frac{1}{2}$ and whose origin is at distance $> \frac{1}{2}$ from 0.
Hence $0 \notin f_1(\sigma)$. $\square$

Theorem 2.2 (Cellular Approximation Theorem)

Let $f: X \longrightarrow Y$ be a map between CW-complexes is homotopic to a cellular map. If f is already cellular on a sub CW-complex A, then we can choose the homotopy to be relative A.

J.H.C. Whitehead (1904–1960).

Whitehead Problem: $0 \to \mathbb{Z} \to A \to B \to 0$ in Ab

If for fixed B, any such ext. splits, is B free?

Shelah: undecidable under ZFC

$\pi_k S^n = 0 \quad k < n$

proof: We proceed by induction, skeleton by skeleton. For X_0, we choose a path from $f(x)$ to a 0-cell y in the path connected component of $f(x) \in Y$, for any $x \in X_0$. This yields a homotopy from $f|_{X_0}$ to a map to cellular on X_0, since $X_0 \subset X$ has the HEP, we obtain a map $f_0 : X \longrightarrow Y$ st. $f|_{X_0}$ is cellular.

Assume now that $f|_{X_{n-1}}$ is cellular and look at one cell e^n, call $\alpha : S^{n-1} \longrightarrow X_{n-1}$ the attaching map:

$$\begin{tikzcd} S^{n-1} \arrow[r, "\alpha"] \arrow[d, hook] & X_{n-1} \arrow[r, "f|_{X_{n-1}}"] \arrow[d, hook] & Y_{n-1} \arrow[d, hook] \\ D^n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X_{n-1} \cup_\alpha e^n \arrow[r, "f"] & Y \end{tikzcd}$$

Since D^n is compact its image in Y meets only finitely many cells.

If all have dim $\leq n$, we're happy, if not let k be the dimension of the largest of them. If f misses a point in $\overset{\circ}{e}{}^k$ we move to the next part of the proof. If not we will use the lemma to make f homotopic to a map f_1 missing a point.

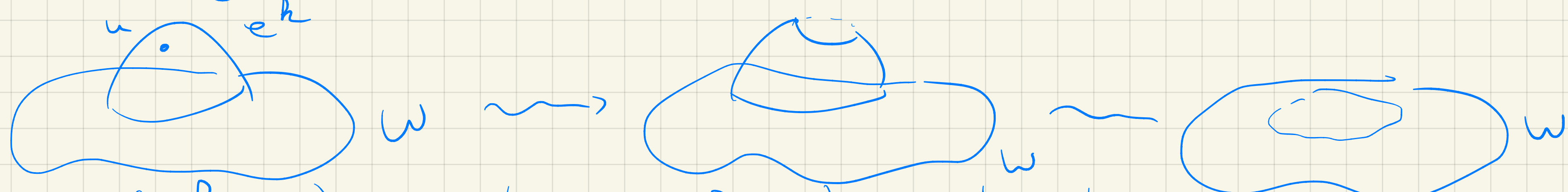
Consider $g : I^n \xrightarrow{\cong} D^n \longrightarrow X_{n-1} \cup_\alpha e^n \xrightarrow{f} Y$

We replace Y be the sub CW-cx made of the cells in the image of g and call it $Z = W \cup e^k$.

By Lemma 2.1 $g \simeq g_1$ relative to $g^{-1}(W)$ and g_1 is PL on a polyhedron $K \subset I^n$ ...

Since $\partial I^n \subset g^{-1}(W)$, so we can extend the homotopy to a homotopy $f \underset{h}{\simeq} f_1$ relative to X_{n-1}. The map f_1 is PL on $K \subset I^n$ and there is an open $U \subset e^k$ s.t. $f_1^{-1}(U) \subset K$.

Because $\dim K = n < k$, f_1 is PL implies that it is not surjective on U as it hits a finite number of affine subspaces in $\mathbb{R}^k \cong \mathring{e}^k$. Let $u \in U$ be a point not in the image.

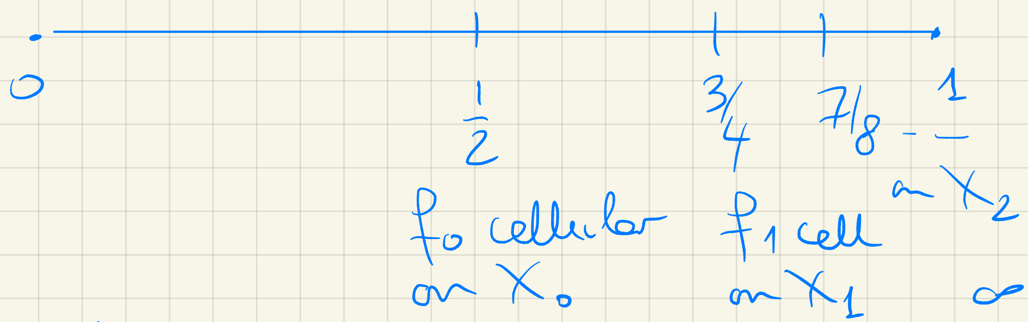


$W \cup (e^k \setminus u)$ admits a deformation retraction onto W

This allows us to construct a homotopy from f to f_1, then to f_2 which misses the whole cell $\mathring{e}_\beta$.
We iterate this process finitely many times to make f cellular on $X_{n-1} \cup_\alpha e^n$.

If X_n has more cells, maybe infinitely many, we apply this process simultaneously on all n-cells.

If $\dim X = \infty$, we concatenate the homotopies making f cellular on X_n, X_{n+1}, etc... in the way indicated by the picture



At $t=1$, f has become cellular on $X = \bigcup_{n=0}^{\infty} X_n$. $\square$

Corollary 2.4

$$\pi_k S^n = 0 \quad \text{for } k < n$$

proof: Any map $S^k \longrightarrow S^n$ is homotopic to a constant map, see Example 1.2. $\square$

Def 2.5 A space X is <u>n-connected</u> if $\pi_k X = 0$ for any $k \leq n$.

0-connected : path-connected

1-connected : simply connected and path-connected

S^n is $(n-1)$-connected

Def 2.6 A pair (X, A) is <u>n-connected</u> if $\pi_k(X, A) = 0$ for $k \leq n$

This means $\pi_k A \cong \pi_k X$ for $k < n$.

$\pi_n A \twoheadrightarrow \pi_n X$ is surjective,