proof: A subbasic open in $\mathrm{map}(Y, Z)$ is of the form $B(K, V)$ for $K \subset Y$ compact, $V \subset Z$ open. For any $f \in B(K, V)$ we construct a neighborhood of the form

$$f \in \bigcap_{\text{finite}} B(K_i, B_i) \quad \text{for } B_i \in \mathcal{B}$$

Since V is open, $V = \bigcup_{\alpha} V_\alpha$, V_α basic open,

$V_\alpha = \bigcap_{j=1}^{n_\alpha} V_{\alpha, j}$, $V_{\alpha,j} \in \mathcal{B}$, $n_\alpha \in \mathbb{N}$, $\forall \alpha$. As $f(K) \subset V$

$\Longrightarrow$ $f^{-1}(V_\alpha)$ cover K in Y. By compacity find $1 \leq i \leq k$

s.t. $f^{-1}(V_{\alpha_i})$ cover K. By Exercise 1, find $K_i \subset f^{-1}(V_{\alpha_i})$

s.t. $K = \bigcup_{i=1}^{k} K_i$. Now $f(K_i) \subset V_{\alpha_i, j}$ $\forall i, \forall j$

This means $f \in \bigcap_{i,j} B(K_i, V_{\alpha_i, j})$ $\square$

Theorem 1.9

Let X, Y, Z be Hausdorff spaces and Y locally compact. Then the exponential law restricts to a Homeo.

$$\mathrm{map}(X \times Y, Z) \approx \mathrm{map}(X, \mathrm{map}(Y, Z))$$

proof: First notice that the set theoretic exponential law

$$Z^{X \times Y} \cong (Z^Y)^X$$

restricts to $\mathrm{map}(X \times Y, Z)$ and sends a map $f: X \times Y \longrightarrow Z$ to a (continuous) map $X \longrightarrow \mathrm{map}(Y, Z)$. Conversely a map $X \xrightarrow{g} \mathrm{map}(Y, Z)$ corresponds to a map $X \times Y \longrightarrow Z$ which is continuous since it is a composition

$$X \times Y \xrightarrow{g} \mathrm{map}(Y, Z) \times Y \xrightarrow{ev} Z.$$

By Lemma 1.6, $\mathcal{B}(K \times L, W)$ form a subbasis of the source for $K \subset X$, $L \subset Y$ compact, $W \subset Z$ open.

By Lemma 1.7 since Z is Hausdorff, so is $\mathrm{map}(Y, Z)$. Hence we can use Lemma 1.8 and get a subbasis for the compact-open topology on the target, namely

$$\mathcal{B}(K, \mathcal{B}(L, W)) \quad \text{for } K, L, W \text{ as above}$$

$\mathcal{B}(L, W)$: subbasis of c-o topology on $\mathrm{map}(Y, Z)$

The exponential law sends $\mathcal{B}(K \times L, W)$ to $\mathcal{B}(K, \mathcal{B}(L, W))$ and vice-versa, so we have indeed a homeo. $\square$

<u>Remark 1.10</u>: • Hausdorff is OK
• locally compact is more restrictive. A CW-complex is locally compact iff every open cell meets only f. many cells (0-dim open cell is a point). In particular $\mathbb{R}P^\infty$, $\mathbb{C}P^\infty$, $\bigvee_1^\infty S^n$ are not locally compact.

§ 2 Categorical considerations (May, Chapter 5)

John Milnor (1931–) advocated for using the category of all spaces having the homotopy type of a CW-complex. He proved $map(X,Y)$ is a CW-cx if X,Y are so and X is compact. It fails if X is not compact in general...

Thus this category is not "Cartesian closed" (exp. law)

Nowadays a commonly used "category of spaces" is that of compactly generated spaces.

Def 2.1. A space X is <u>weak Hausdorff</u> if any map $g: K \longrightarrow X$ with K compact has closed image.

Def 2.2 A subspace $A \subset X$ is <u>compactly closed</u> if $g^{-1}(A)$ in K for g as in 2.1 is closed in K.

If X is weak Hausdorff, A is compactly closed means $A \cap L$ is closed in X for any compact $L \subset X$.

Def 2.3 A <u>k-space</u> X is a space st. every compactly closed subspace A is closed. A weak Hausdorff k-space is a <u>compactly generated</u> space

Hurewicz (1904-1956)

McCord (1969) noticed the Cartesian-closed property.

⚠ The product of X and Y is not $X \times Y$, but $k(X \times Y)$ its <u>k-ification</u> where all compactly closed subspaces are turned into closed ones. Idem for map...

§ 3 Pointed mapping spaces (Strom, Chapter 3)

All spaces are pointed, so X will mean (X, x_0), and ______ are Hausdorff and locally compact.

Def 3.1 Let X, Y be pointed spaces. The <u>pointed mapping space</u> $\mathrm{map}_*(X,Y)$ is the subspace of $\mathrm{map}(X,Y)$ of $f : X \to Y$ s.t. $f(x_0) = y_0$. The base point is $c_{y_0} : X \to Y$, $x \mapsto y_0$.

Recall that the <u>smash product</u> $X \wedge Y = X \times Y / X \vee Y$.

Lemma 3.2

Let
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "i"] & D \end{tikzcd}$$
be a pushout square of Hausdorff and locally compact spaces. Then
$$\begin{tikzcd} \mathrm{map}(D,Y) \arrow[r, "i^*"] \arrow[d, "h^*"'] & \mathrm{map}(C,Y) \arrow[d, "g^*"] \\ \mathrm{map}(B,Y) \arrow[r, "f^*"] & \mathrm{map}(A,Y) \end{tikzcd}$$
is a pullback square.

proof: We check the universal property: Consider

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!\delta\ ?"] \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "\beta"'] & & \\
& \mathrm{map}(D,Y) \arrow[r] \arrow[d] \arrow[ul, phantom, "①"] & \mathrm{map}(C,Y) \arrow[d, "g^*"] \\
& \mathrm{map}(B,Y) \arrow[r, "f^*"'] & \mathrm{map}(A,Y)
\end{tikzcd}$$

By adjunction we find a diagram

$$\begin{tikzcd}
X \times A \arrow[r, "1\times f"] \arrow[d, "1\times g"'] & X \times B \arrow[d, "1\times h"] \arrow[ddr, bend left, "b=a(\beta)"] & \\
X \times C \arrow[r, "1\times i"'] \arrow[drr, bend right, "k=a(x)"'] & X \times D \arrow[dr, dashed, "\exists!d"] & \\
& & Y
\end{tikzcd}$$

which commutes by naturality, for example $b \circ (1 \times f) = a(f^* \circ \beta)$

Since $X \times -$ is a left adjoint so it preserves pushout, therefore d exists and is unique (and ① and ② commute). Choose $\delta = b(d)$ so ①, ② commute. Since $a(\delta) = d$, δ is unique. $\square$

Corollary 3.3

Let $A \subset X$ and X/A is Hausdorff and locally compact. Then $\mathrm{map}_*(X/A, Y)$ is homeomorphic to the subspace of $\mathrm{map}(X, Y)$ of $f : X \to Y$ st $f(A) = y_0$.

<u>proof:</u> X/A has $[a]$ as basepoint for any $a \in A$.

The pushout

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & X \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner", near end] & X/A \end{tikzcd}$$

induces by Lemma 3.2 a pullback square

$$\begin{tikzcd} \mathrm{map}(X/A, Y) \arrow[r, "i^*"] \arrow[d] \arrow[dr, phantom, "\lrcorner", near start] & \mathrm{map}(*, Y) \simeq Y \ni y \arrow[d] \\ \mathrm{map}(X, Y) \arrow[r] & \mathrm{map}(A, Y) \end{tikzcd}$$

$f \longmapsto f|_A$, $\quad y \longmapsto c_y$. So $\mathrm{map}(X/A, Y)$ is a subspace of $\mathrm{map}(X, Y) \times Y$. Pointed maps correspond to pairs (f, y_0).

Then $\mathrm{map}_*(X/A, Y)$ is thus identified with a subspace of $\mathrm{map}(X, Y) \times y_0$ of (f, y_0) st. $f|_A = c_{y_0}$. $\square$

Corollary 3.4

$\mathrm{map}_*(X \wedge Y, Z)$ is homeomorphic to the subspace of $\mathrm{map}(X \times Y, Z)$ of maps $f : X \times Y \longrightarrow Z$ s.t. $f|_{X \vee Y} \equiv c_{z_0}$

Theorem 3.5

We have a homeo $\mathrm{map}_*(X \wedge Y, Z) \approx \mathrm{map}_*(X, \mathrm{map}_*(Y, Z))$

proof: We restrict the homeo $\mathrm{map}(X \times Y, Z) \approx \mathrm{map}(X, \mathrm{map}(Y, Z))$ to $\mathrm{map}_*(X \wedge Y, Z)$ by Corollary 3.4. The image $a(f)$ of a map $f : X \times Y \longrightarrow Z$ s.t. $f|_{X \vee Y} \equiv c_{z_0}$ is a map $a(f) : X \longrightarrow \mathrm{map}(Y, Z)$

for any $x \in X$,

$$x \longmapsto a(f)(x) = f(x, \_) : y_0 \longmapsto z_0$$

$$x_0 \longmapsto a(f)(x_0) = f(x_0, \_) = c_{z_0}$$

Hence $a(f)$ belongs to $\mathrm{map}_*(X, \mathrm{map}_*(Y, Z))$

Conversely the inverse homeo b from Section 1 sends $\mathrm{map}_*(X, \mathrm{map}_*(Y, Z))$ to $\mathrm{map}_*(X \wedge Y, Z)$. $\square$

1996, PARIS, Cagliari

Corollary 3.6

(i) The smash product $X \wedge \_$ converts pushouts into pushouts
(ii) The pointed map$_*$ $\mathrm{map}_*(X, \_)$ — pullbacks into pullbacks
(iii) The ——— $\mathrm{map}_*(\_, Y)$ — pushouts into pullbacks

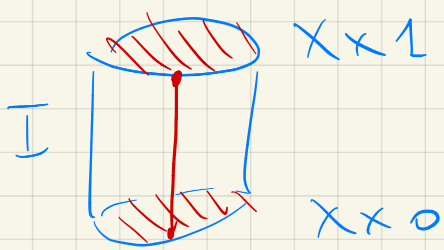
proof: (i) $X \wedge \_$ left adjoint
(ii) $\mathrm{map}_*(X, \_)$ right adjoint
(iii) like Lemma 3.2 $\square$

Def 3.7 The loop space $\Omega X = \mathrm{map}_*(S^1, X)$

Elements of ΩX are based loops in X.

Def. 3.8 The reduced suspension $SX = S^1 \wedge X$

Note: $SX = (I/_{0\sim 1}) \wedge X = I \times X \Big/ \begin{array}{ll} (0,x) \sim (1,x) & \forall x \\ (t,x_0) \sim (0,x_0) & \forall t \\ (0,x) \sim (0,x_0) & \forall x \end{array}$



a "pointed suspension"

Proposition 3.9

$$\mathrm{map}_*(X, \Omega Y) \simeq \mathrm{map}_*(SX, Y)$$

$\square$

$$S \dashv \Omega$$

In particular $[X, \Omega Y]_* \cong [SX, Y]_*$ (take π_0).