Chapter I. The compact-open topology

Aim: upgrade the set of (continuous) maps $f: X \to Y$ between spaces X and Y to a space of maps $\mathrm{map}(X,Y)$ by defining a topology on this set.

Thus the category of spaces is enriched over Top.
Also in Top_*.

§1. The compact-open topology

Hatcher, Appendix A

Here X, Y, Z are spaces, all maps are continuous.

Def 1.1 The set of all maps $f: X \to Y$ is given the structure of a space $\mathrm{map}(X,Y)$ with the compact-open topology whose subbasis is
$$B(K,U) = \{ f: X \to Y \mid f(K) \subset U \}$$
for K compact in X, U open in Y.

Therefore a basis for this topology is given by finite intersections of $B(K_i, U_i)$, $1 \leq i \leq n$:

$$B(K_\bullet, U_\bullet) = \{ f : X \to Y \mid f(K_i) \subset U_i, \forall 1 \leq i \leq n \}.$$

An arbitrary open subset of $map(X, Y)$ is an arbitrary union of such $B(K_\bullet, U_\bullet)$.

Remark 1.2: We will mostly work with finite CW-complexes as source of $map(X, Y)$. If Y is a CW-cx then $map(X, Y)$ has the homotopy type of a CW-cx, but this not true if X is an infinite CW-cx in general.

Examples 1.3 ① $X = *$, $map(*, Y) \approx Y$ (Homeo)

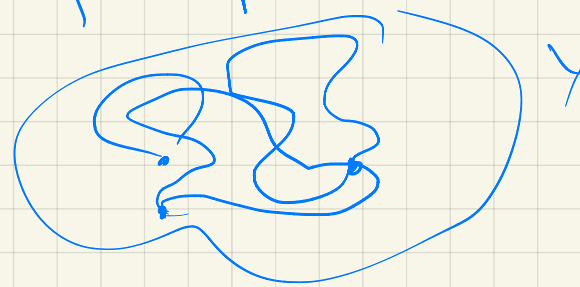
② $X = \{1, \dots, n\}$ discrete, $map(X, Y) \approx Y^n$

Compact-open topology corresponds to product topology.

③ $X = I = [0;1]$, $map(I, Y)$ is the space of paths in Y. One can choose a nicer basis here: choose a partition $0 = t_0 < t_1 < \dots < t_n = 1$, $U_1, \dots, U_n \subset Y$ open

$$\{ f : I \longrightarrow Y \mid f[t_{i-1}, t_i] \subset U_i \quad \forall\ 1 \le i \le n \}$$

(4) $X = S^1$, $\mathrm{map}(S^1, Y) = \Lambda Y$ is the space of <u>free loops</u> (free: base point of loop is arbitrary)



<u>Reminder 1.4</u> Exponential law.

$\mathrm{mor}_{Set}(X, Y) = Y^X$. There is a <u>natural</u> bijection in <u>Set</u>

$$Z^{X \times Y} \cong (Z^Y)^X$$

$$b(\varphi) = f : X \times Y \longrightarrow Z \longleftrightarrow a(f) = \varphi : X \longrightarrow Z^Y$$
$$x \longmapsto f(x, \_)$$

In fact $X \times \_$ is left adjoint to Z^{-}.

Inside $Y^X \supset \mathrm{map}(X, Y)$. We ask if this adjunction is compatible with the compact-open topology: When do we have $\mathrm{map}(X \times Y, Z) \approx \mathrm{map}(X, \mathrm{map}(Y, Z))$?

This isn't always the case, we will impose restrictions.

Proposition 1.5

(a) Let X be locally compact, then $ev : \mathrm{map}(X,Y) \times X \longrightarrow Y$, $(f, x) \longmapsto f(x)$ is continuous

(b) If Y is locally compact, a map $f : X \times Y \longrightarrow Z$ is continuous iff $a(f) : X \longrightarrow \mathrm{map}(Y, Z)$ is continuous.

proof (a) Let $U \ni f(x)$ be open in Y. Since f is continuous, $f^{-1}(U)$ is an open neighborhood of x. By assumption X is locally compact, so there is $x \in K \subset f^{-1}(U)$, K compact, i.e. $f(K) \subset U$. We choose $B(K, U)$ in $\mathrm{map}(X, Y)$ and K as neighborhood of $x \in X$.

$$ev(\underbrace{B(K,U) \times K}_{\text{neighborhood of } (f,x) \text{ in } ev^{-1}(U)}) \subset U.$$

(b) Notice first that if $f : X \times Y \longrightarrow Z$ is a map, then $f|_{x \times Y}$ is also continuous for any $x \in X$. Hence

$$a(f) : X \longrightarrow Z^Y \quad \text{lands actually in } \mathrm{map}(Y, Z)$$
$$x \longmapsto f|_{x \times Y}$$

Next show $a(f)$ is continuous by looking at the preimage of $B(L, W)$ for $L \subset Y$ compact, W open in Z.

$$a(f)^{-1}(B(L, W)) = \{x \in X \mid f(x \times L) \subset W\}.$$

Since W is open, $f^{-1}(W)$ is open in $X \times Y$ and contains $x \times L$. Let us choose an open subset $U \times V$ for the product topology in $X \times Y$, I claim we can assume it contains $x \times L$ because L is compact.

We have shown that every $x \in a(f)^{-1}(B(L, W))$ admits an open neighborhood in this preimage.

Last part: Assume $\varphi : X \longrightarrow \mathrm{map}(Y, Z)$ is continuous. We prove $b(\varphi)$ is continuous.

$$b(\varphi) : X \times Y \xrightarrow{\varphi \times 1} \mathrm{map}(Y, Z) \times Y \xrightarrow{ev} Z$$
$$(x, y) \longmapsto (\varphi(x), y) \longmapsto \varphi(x)(y)$$

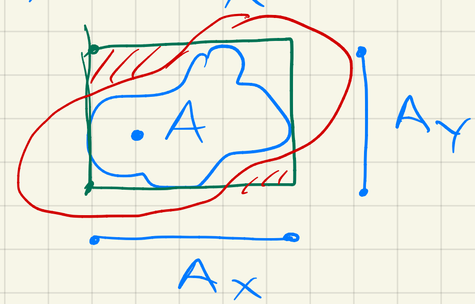
So $b(\varphi)$ is continuous because $\varphi \times 1$ is so, and ev by ⓐ as Y is locally compact. $\square$

Lemma 1.6

Let X, Y be Hausdorff spaces. Then a subbasis for $\mathrm{map}(X \times Y, Z)$ is given by $B(K \times L, W)$ where K compact in X, L compact in Y, W open in Z.

<u>proof</u>: Let $A \subset X \times Y$ be compact, $A_X = \pi_X(A)$, $A_Y = \pi_Y(A)$ are compact (projections of a compact). For $f \in B(A, W)$, in principle $f \notin B(A_X \times A_Y, W)$ (////)

$f^{-1}(W)$



For any $(x,y) \in A$, $(x,y) \in \underbrace{A_X \times A_Y \cap f^{-1}(W)}_{D}$

Chose $(x,y) \in U_x \times V_y \subset D$

(since $A_X \times A_Y$ is compact and Hausdorff, it's normal and we can separate (x,y) from the complement of $f^{-1}(W)$).

Even better we separate x from the complement of U_x and y from the complement of V_y so as to obtain $x \in U \subset \overline{U} \subset U_x$ and $y \in V \subset \overline{V} \subset V_y$.

The compact subspace $\overline{U} \times \overline{V}$ cover A, we extract a finite cover $\overline{U_i} \times \overline{V_i}$, $1 \leq i \leq n$. Then

$f \in B(\overline{U_i} \times \overline{V_i}, W)$ since $\overline{U_i} \times \overline{V_i} \subset A$.

Finally $f \in \bigcap_{i=1}^{n} B(\overline{U_i} \times \overline{V_i}, W)$. $\square$

Notice that $a(\_)$ sends $B(K \times L, W)$ to $B(K, B(L, W))$

Lemma 1.7

If Z is Hausdorff $\mathrm{map}(Y, Z)$ is so.

exercise

Lemma 1.8

If Z is Hausdorff, $\mathcal{B}$ is a subbasis of its topology then $B(K, \mathcal{B})$ for $K \subset Y$ compact form a subbasis for the compact-open topology on $\mathrm{map}(Y, Z)$