EPFL - Fall Semester 2022-2023	Jérôme Scherer
Homotopy Theory	**Final Examen**
Mathematics	16 January 2023
Duration : 180 minutes	Number of points : 60

Name		First name	
Signature		Sciper	

Justifications and explanations based on the theory seen in class are required in each exercise. Indicate not only that such and such result has been proved in class, but also what this result says (unless it has a clear name like Hurewicz Theorem or cellular approximation Theorem).
Since Exercise 1 is about theoretical results seen in class, we expect detailed answers based on facts we have seen before studying the chapter in question.

The following table is reserved for correction.

Ex. 1 (15 pts)		Ex. 2 (15 pts)		Ex. 3 (15 pts)	
Ex. 4 (15 pts)		Written Ex.			
Total (60 pts)		Oral Ex.		**Final Grade**	

Question 1.The dual Puppe sequence. (15 points) Let $(X; x_0)$ be a path connected pointed space, $I = [0, 1]$ be the unit interval (with 0 as a base point when needed), and $map(I, X)$ be the space of all unpointed paths $\alpha\colon I \to X$. We write $ev\colon map(I, X) \to X \times X$ for the evaluation at 0 and 1, $ev(\alpha) = (\alpha(0); \alpha(1))$.

(a) Explain why the evaluation map ev is a fibration.

(b) Prove that $ev_1\colon map_*(I, X) \to X$ is a fibration as well, where $ev_1(\alpha) = \alpha(1)$ for any *pointed* path $\alpha\colon I \to X$.

(c) Let Y be a space and $f\colon Y \to X$ a map. Explain how to factor the map f into a homotopy equivalence followed by a fibration $Y \xrightarrow{\simeq} Y' \xrightarrow{p} X$. Prove that p is a fibration (the proof of the homotopy equivalence is not required).

(d) Let $y_0 \in Y$ be a base point for Y and assume f is a pointed map. Show that the mapping fiber $F(f)$, pullback of the diagram $Y \xrightarrow{f} X \xleftarrow{ev_1} map_*(I, X)$ is homotopy equivalent to the homotopy fiber $Fib(f)$. Provide an explicit map between $F(f)$ and $Fib(f)$ and prove it is a homotopy equivalence. The homotopy fiber of f is by definition $p^{-1}(x_0)$, the preimage of the base point under the fibration p constructed in (c).

(e) Show that the homotopy fiber of $F(f) \to Y$ is homotopy equivalent to ΩX.

Question 2. Fibrations with complex projective spaces. (15 points) The unit circle $S^1 \subset \mathbb{C}$ acts by right multiplication on the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ via the formula

$$(z_0; z_1; \dots z_n) \cdot z = (z_0 \cdot z; z_1 \cdot z; \dots z_n \cdot z)$$

for $(z_0; z_1; \dots z_n) \in S^{2n+1}$ and $z \in S^1$. We admit that the quotient maps $S^{2n+1} \to \mathbb{C}P^n$, under the action of the unit circle S^1, are fibrations.

(a) Compute the first two non-trivial homotopy groups of $\mathbb{C}P^n$ for any $n \geq 1$.

(b) Show that the inclusion $\mathbb{C}P^n \to \mathbb{C}P^{n+1}$ induces an isomorphism on π_k for all $k \leq 2n$.

(c) Let F be the homotopy fiber of the inclusion $\mathbb{C}P^n \to \mathbb{C}P^{n+1}$. Show that F is $2n$-connected and compute $\pi_{2n+1}F$.

(d) Prove that $\Omega\mathbb{C}P^n$ is homotopy equivalent to the product $S^1 \times \Omega S^{2n+1}$.

Question 3. Moore spaces. (15 points) Let $n \geq 2$ and A be a finitely generated abelian group. A *Moore space* of type $M(A,n)$ is an $(n-1)$-connected CW-complex X such that $H_k(X;\mathbb{Z}) = 0$ for all $k \neq 0, n$ and $H_n(X;\mathbb{Z}) \cong A$.

(a) Show that a finite wedge of spheres of the same dimension n is a Moore space and identify its type.

(b) Construct a Moore space X of type $M(A,n)$ for any $n \geq 2$ and any finitely generated abelian group A.

(c) Let X be the Moore space from (b). Compute $\pi_k X$ for all $k \leq n$.

(d) Using the relative Hurewicz Theorem prove that a map $f\colon Y \to Z$ between simply connected spaces that induces an isomorphism on all integral homology groups is a weak homotopy equivalence.

(e) Prove that any Moore space of type $M(A,n)$ is homotopy equivalent to the space X from (b).

Question 4. Cubical diagrams. (15 points) Let I be the "pushout category" $1 \leftarrow 0 \rightarrow 2$ consisting of three objects and only two non-identity morphisms as illustrated above. This can be seen alternatively as the category associated to the poset of non-maximal subsets of $\{1, 2\}$, namely $\emptyset$ (which we write 0), and singletons $\{1\}$ and $\{2\}$. The partial order is given by inclusion. Let J be the category of non-maximal subsets of $\{1, 2, 3\}$. We write k for the singleton $\{k\}$ and jk for the subset $\{j, k\}$.

(a) Draw the category J as a cube with a missing vertex.

(b) Let C be a category having all colimits. Let $F\colon J \to \mathsf{C}$ be a diagram of shape J and let C denote its colimit. Show that there is a morphism $C(0) := \operatorname{colim}(F(2) \leftarrow F(0) \rightarrow F(1)) \rightarrow F(12)$ and a morphism $C(0) \rightarrow C(3) := \operatorname{colim}(F(23) \leftarrow F(3) \rightarrow F(13))$.

(c) Let D be the colimit of the pushout diagram $C(3) \leftarrow C(0) \rightarrow F(12)$ (using the same notation as above). Show that there is a morphism $C \to D$ and another one $D \to C$.

(d) With the same notation as above show that the composition $C \to D \to C$ is the identity. We will admit that the other composition $D \to C \to D$ is also the identity. Conclude that the cubical colimit C can be computed as the "pushout of pushouts" D.

(e) From now on C is the category of (unpointed) topological spaces. We are given a recipe : By using the mapping cylinder turn the following maps into cofibrations : $F(0) \to F(1)$ into $QF(0) = F(0) \hookrightarrow QF(1) \xrightarrow{\simeq} F(1)$; $F(3) \to F(13)$ into $QF(3) = F(3) \hookrightarrow QF(13)$. Set $QF(2) = F(2)$ and $QF(23) = F(23)$ and turn the map $\operatorname{colim}(QF(1) \leftarrow QF(0) \rightarrow QF(2)) \rightarrow F(12)$ into a cofibration defining a space $QF(12)$; explain why this defines a diagram QF and consider its colimit. Show that this provides a homotopy invariant version of the cubical colimit, i.e. that any natural transformation of cubical diagram $F \to F'$ such that $F(S) \to F'(S)$ is a homotopy equivalence for all objects $S \in J$ induces a homotopy equivalence $\operatorname{colim}_J QF \to \operatorname{colim}_J QF'$. You can use the homotopy invariance of homotopy pushouts as studied in the course.

(f) We write $\operatorname{hocolim}_J F$ for the space $\operatorname{colim}_J QF$ defined above. Identify the homotopy type of the homotopy colimit of the diagram F defined by $F(0) = S^0$, $F(i) = D^1$ for all $1 \leq i \leq 3$, and $F(jk) = S^1$ for all $j \neq k$. All maps are inclusions, $F(0)$ is the boundary of the 1-disc $F(i)$ for all i, and $F(j)$, respectively $F(k)$, is the bottom, respectively top, hemicycle in $F(jk)$ for all $j \neq k$.