

EPFL - Fall Semester 2021-2022
Homotopy Theory
MATHEMATICS
Duration : 180 minutes

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Final Examen
17 January 2022
Number of points : 60

Name		First name	
Signature		Sciper	

Justifications and explanations based on the theory seen in class are required in each exercise.

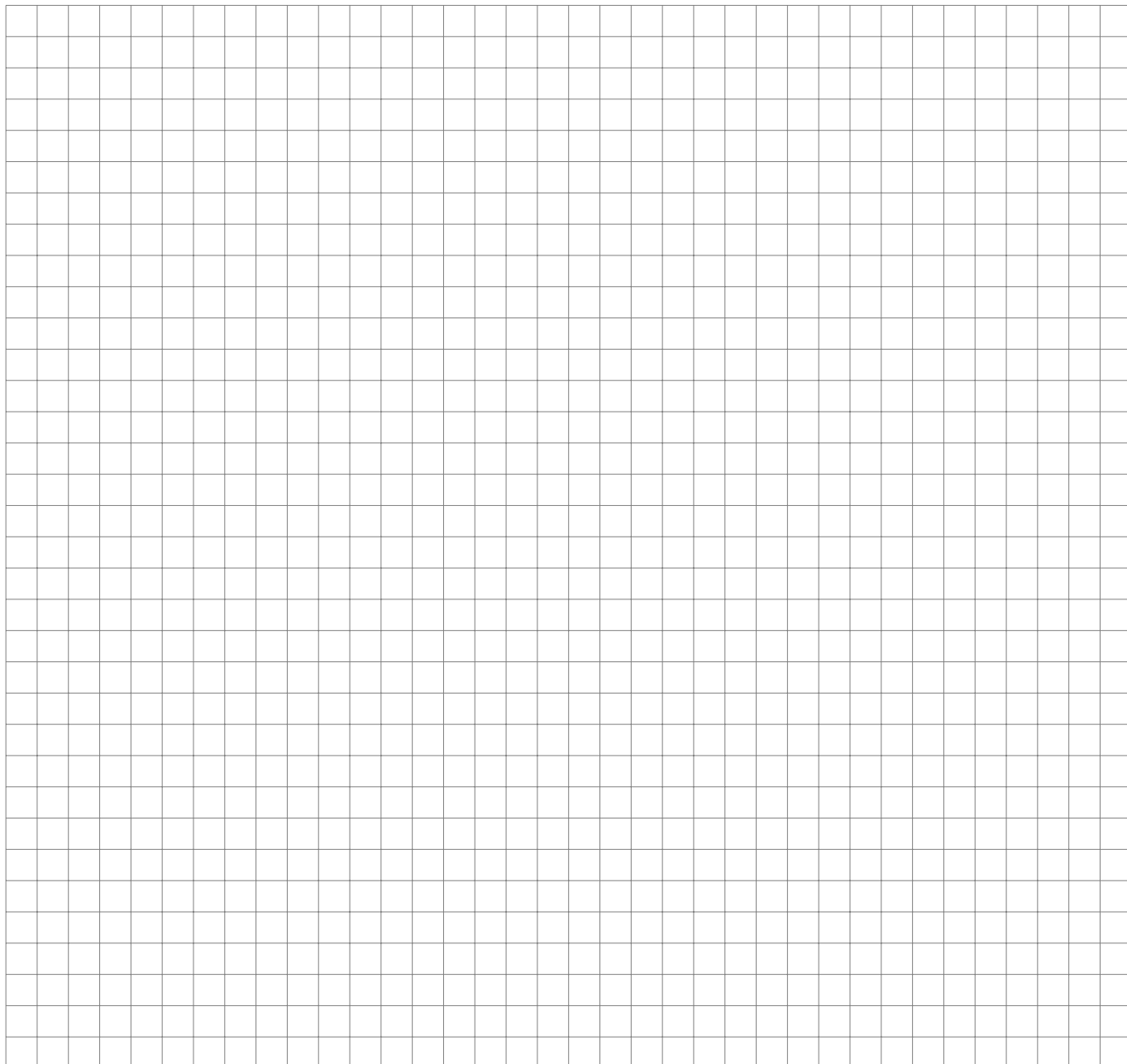
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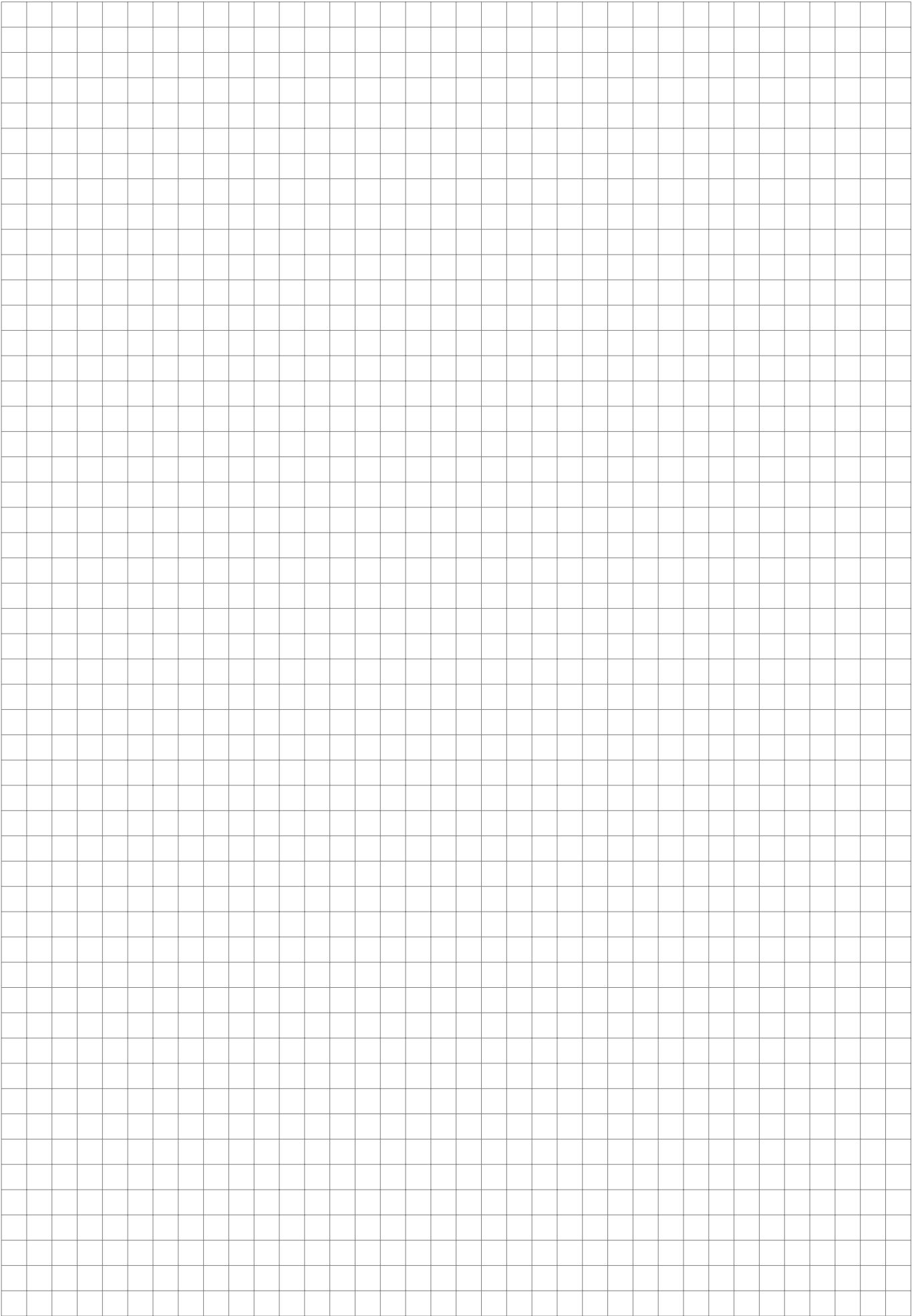
Ex. 1 (15 pts)		Ex. 2 (15 pts)		Ex. 3 (15 pts)	
Ex. 4 (15 pts)		Written Ex.			
Total (60 pts)		Oral Ex.		Final Grade	

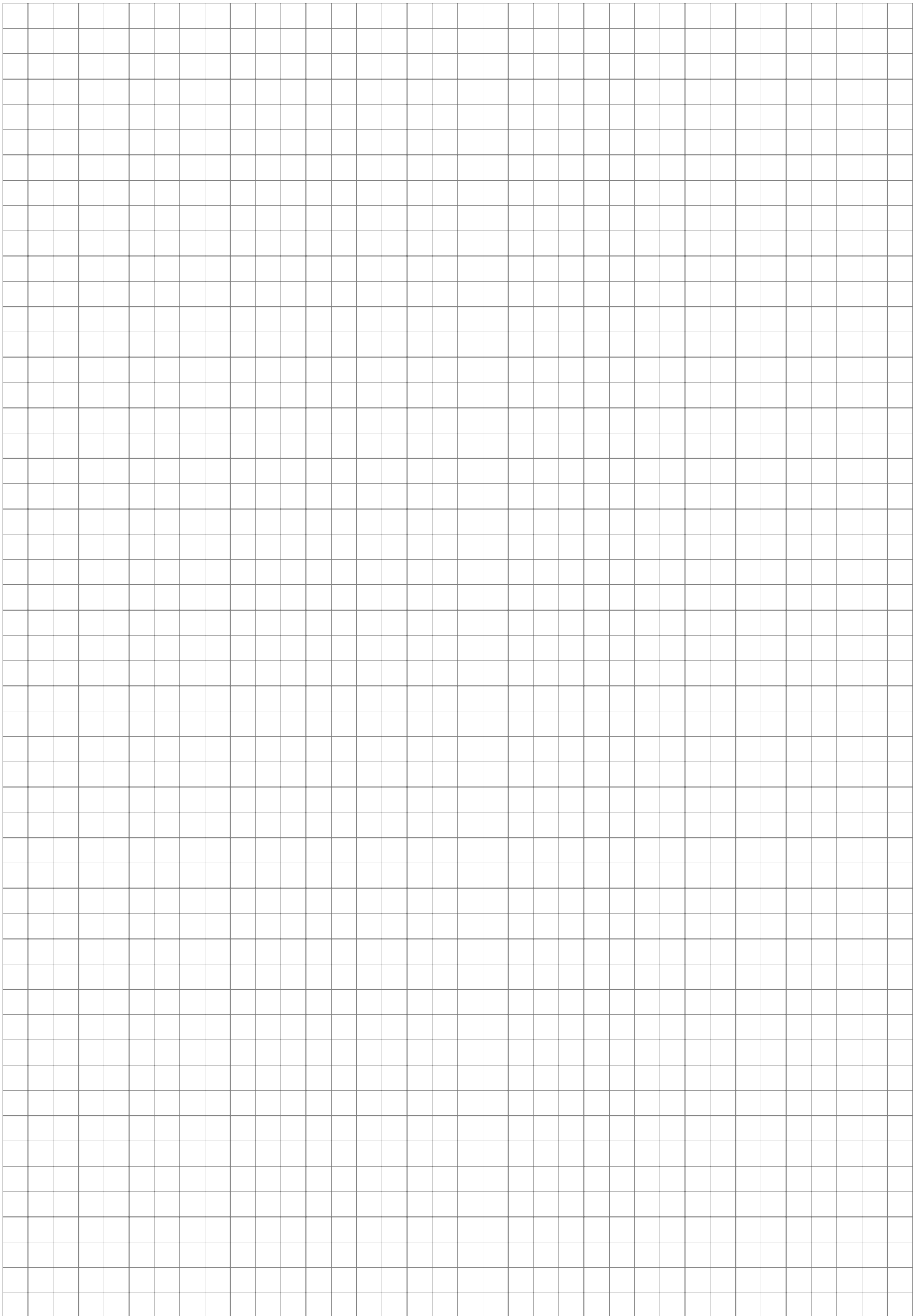
Question 1, a computation. (15 points) The final goal of this exercise is the computation of the set of pointed homotopy classes of maps $[\mathbb{R}P^2, S^2]$. We use the notation c for the constant map, $p: S^2 \rightarrow \mathbb{R}P^2$ for the quotient by the antipodal action of C_2 , and $q: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2/\mathbb{R}P^1$ for the quotient map. Apply the following strategy.

- Identify the first six terms of the Puppe sequence for the degree 2 map $f: S^1 \rightarrow S^1$.
- Obtain a long exact sequence of sets of pointed homotopy classes of maps into S^2 .
- Perform then a careful analysis of exactness (sometimes of sets, not of groups!) so as to identify one representative $\mathbb{R}P^2 \rightarrow S^2$ for each homotopy class, namely the images under q^* of any self-map of S^2 of even, respectively odd, degree.
- Show that there does not exist any map $f: \mathbb{R}P^2 \rightarrow S^2$ such that $f \circ p$ is homotopic to the identity. Find then a space X such that $\pi_n X \cong \pi_n \mathbb{R}P^2$ for all $n \geq 0$ but is not homotopy equivalent to $\mathbb{R}P^2$.

Note. All results about the real projective spaces, quotient maps from the Topology class, the CW-structure of real projective spaces from the Algebraic Topology class can be used. In particular you can use the fact that $q \circ p$ has degree zero.

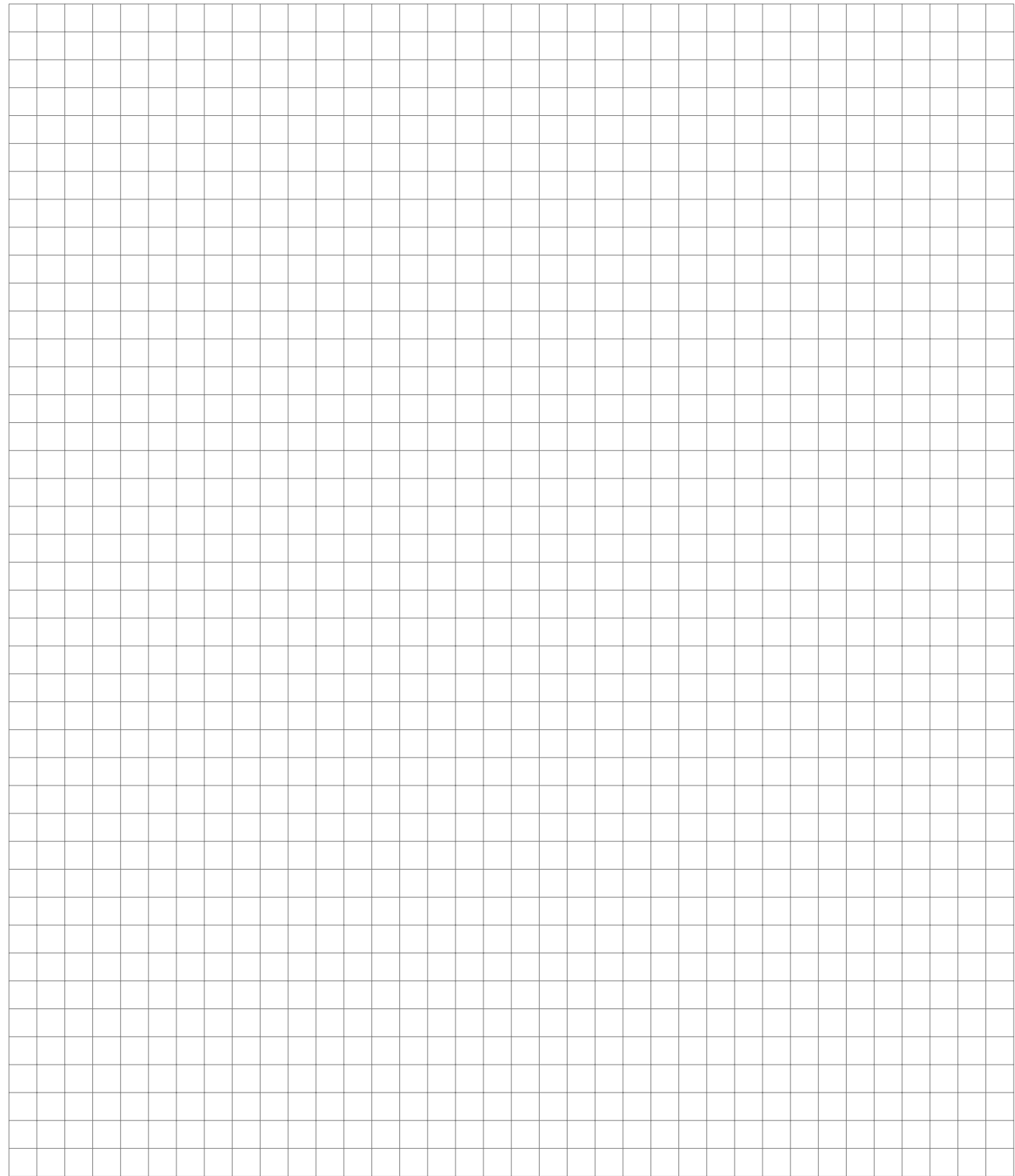


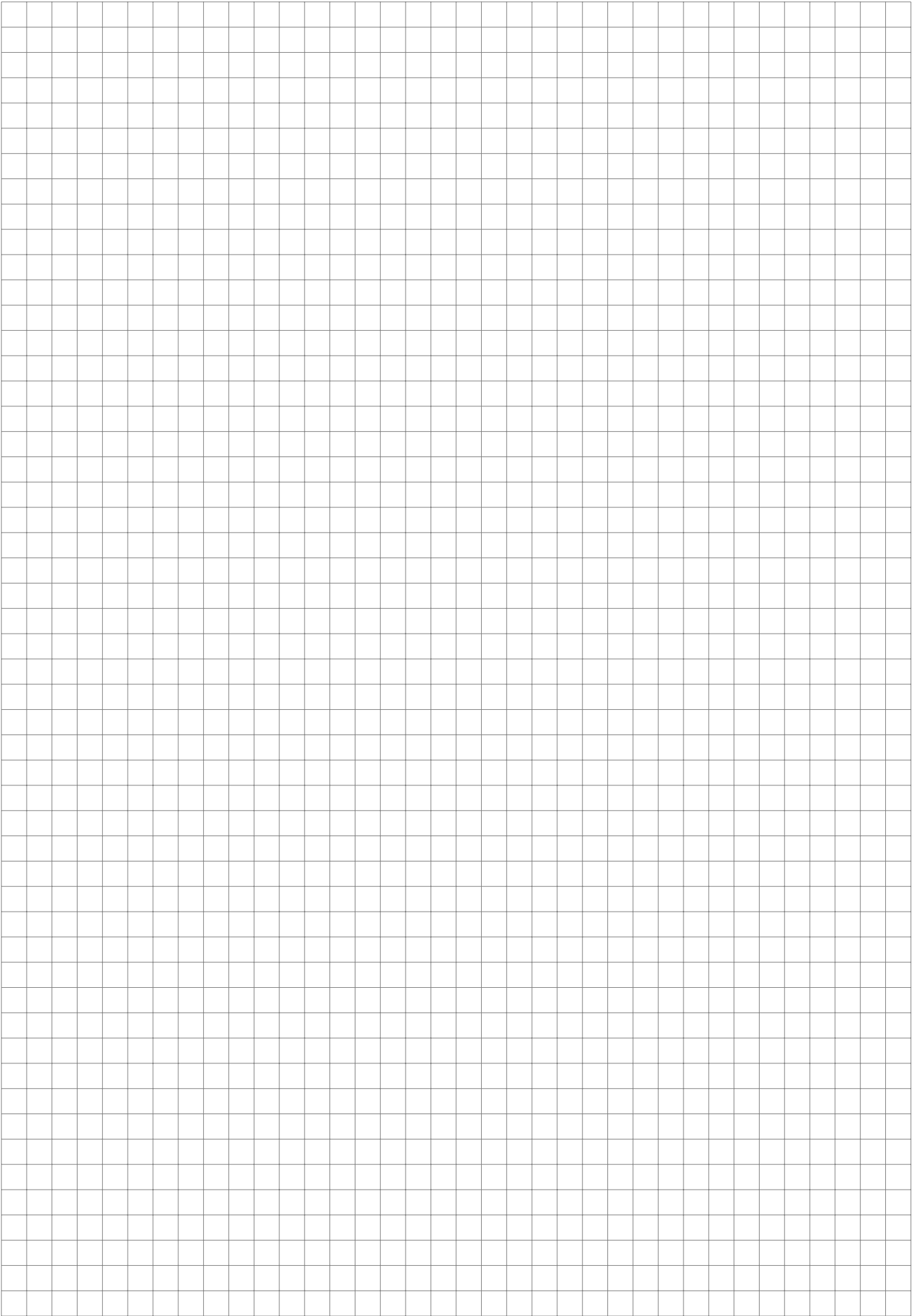


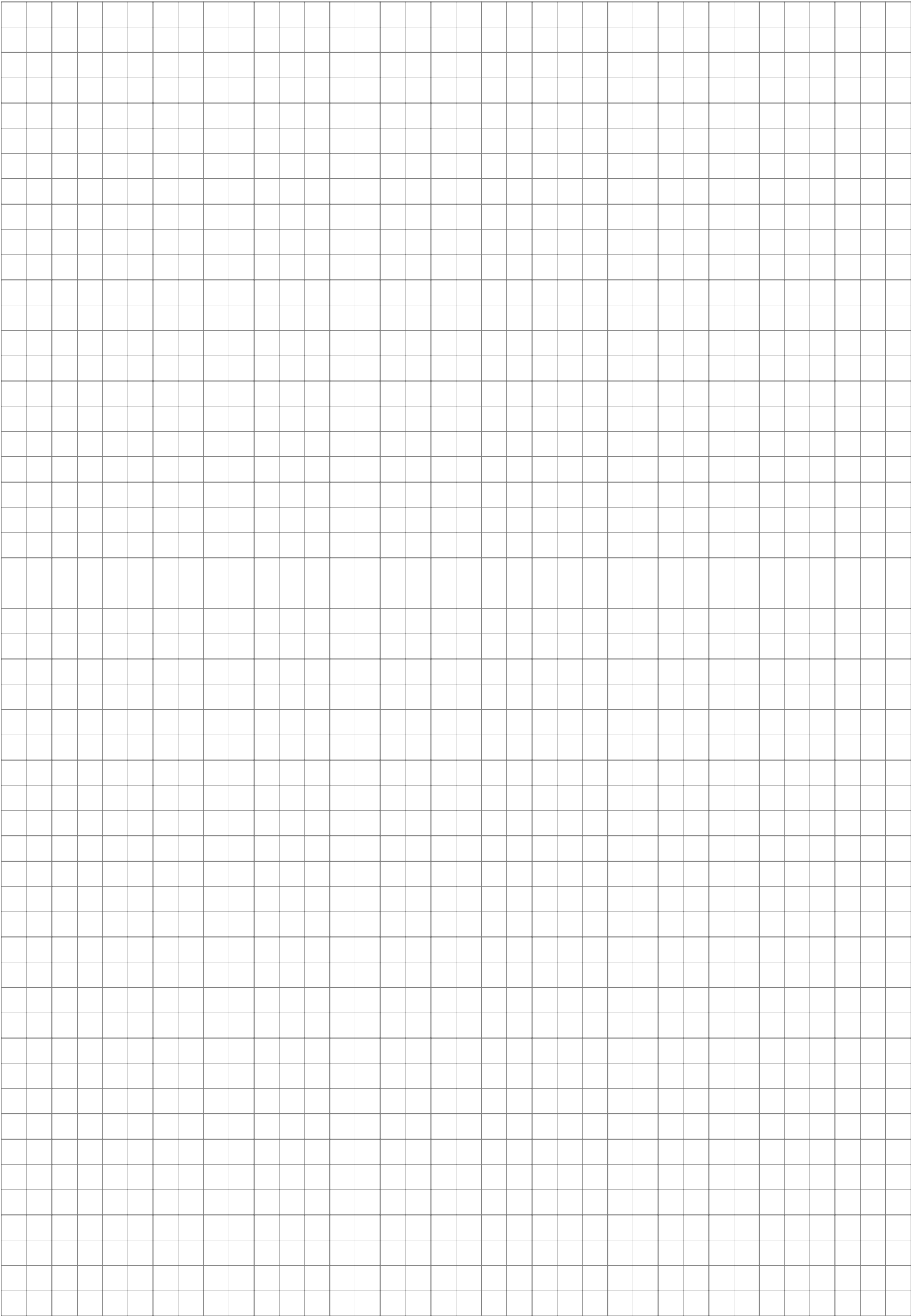


Question 2, a complement to the Hurewicz Theorem. (15 points) Let $n \geq 2$ and X be a pointed $(n - 1)$ -connected space. The final goal of this exercise is to prove that the Hurewicz homomorphism $Hu: \pi_{n+1}X \rightarrow H_{n+1}(X; \mathbb{Z})$ is surjective. You will first explain how to reduce the proof to the case of a CW-complex of dimension $n + 1$. Show by an example that Hu is not an isomorphism in general.

Hint. To prove surjectivity of Hu for a CW-complex X of dimension $n + 1$ we suggest to fix a cycle $\sigma \in C_{n+1}^{cell}(X)$ and represent a preimage under Hu as a pushout of a natural transformation of diagrams from $D^{n+1} \leftarrow S^n \rightarrow D^{n+1}$ to the pushout diagram defining the CW-complex X .







Question 3, Two homotopy pushouts. (15 points) Let (A, a_0) , (B, b_0) , (C, c_0) , and (D, d_0) be well-pointed, locally compact, and Hausdorff spaces. Write $c: A \rightarrow C$ and $d: B \rightarrow D$ for the constant maps to the respective base points, and define the half-smash $A \ltimes D = (A \times D)/(A \times d_0)$.

- (a) Explain why $A \times (-)$ converts homotopy pushout squares into homotopy pushout squares.
 (b) Identify the homotopy pushout of the diagram

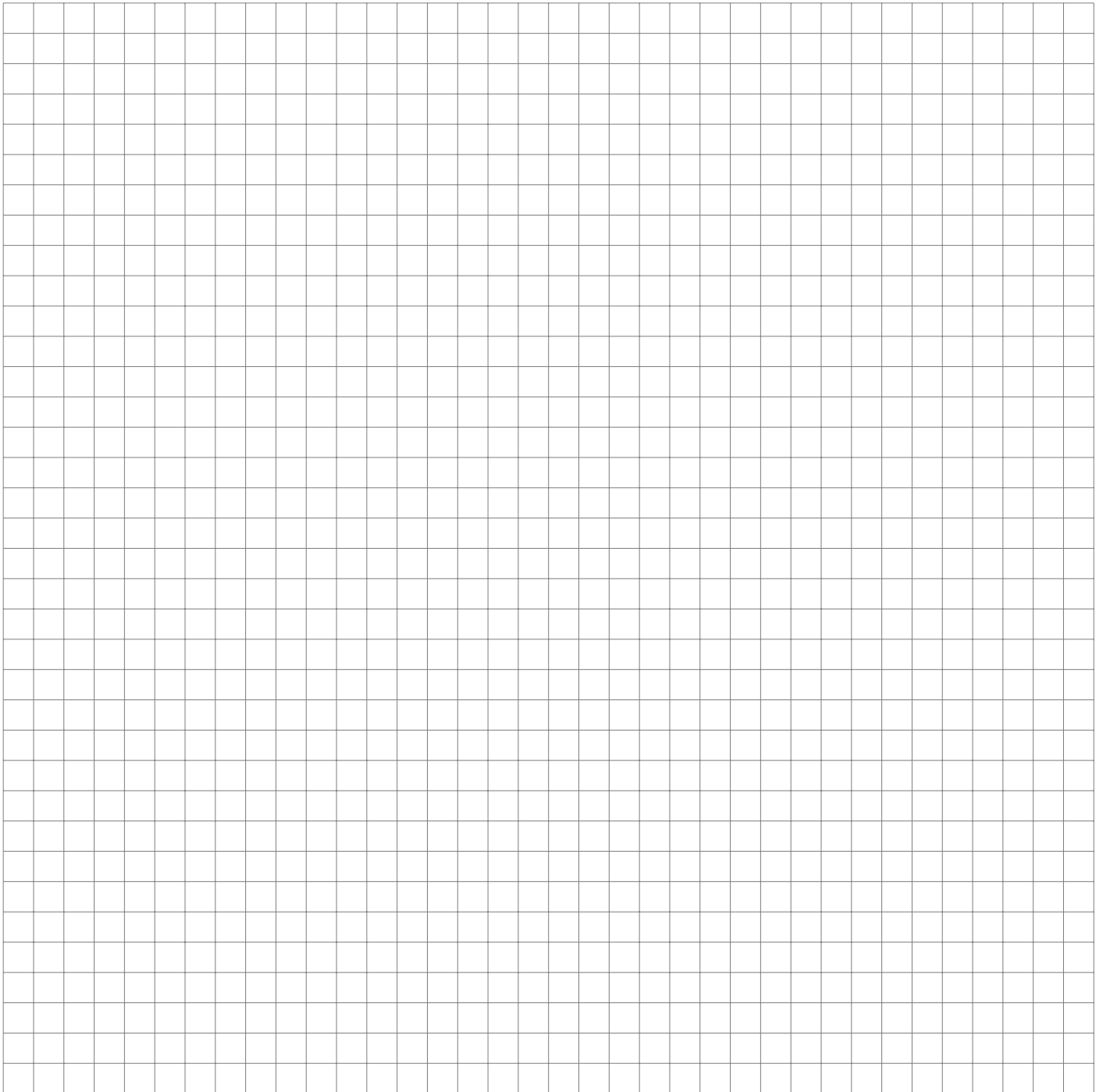
$$A \times D \xleftarrow{id_A \times d} A \times B \xrightarrow{c \times id_B} C \times B$$

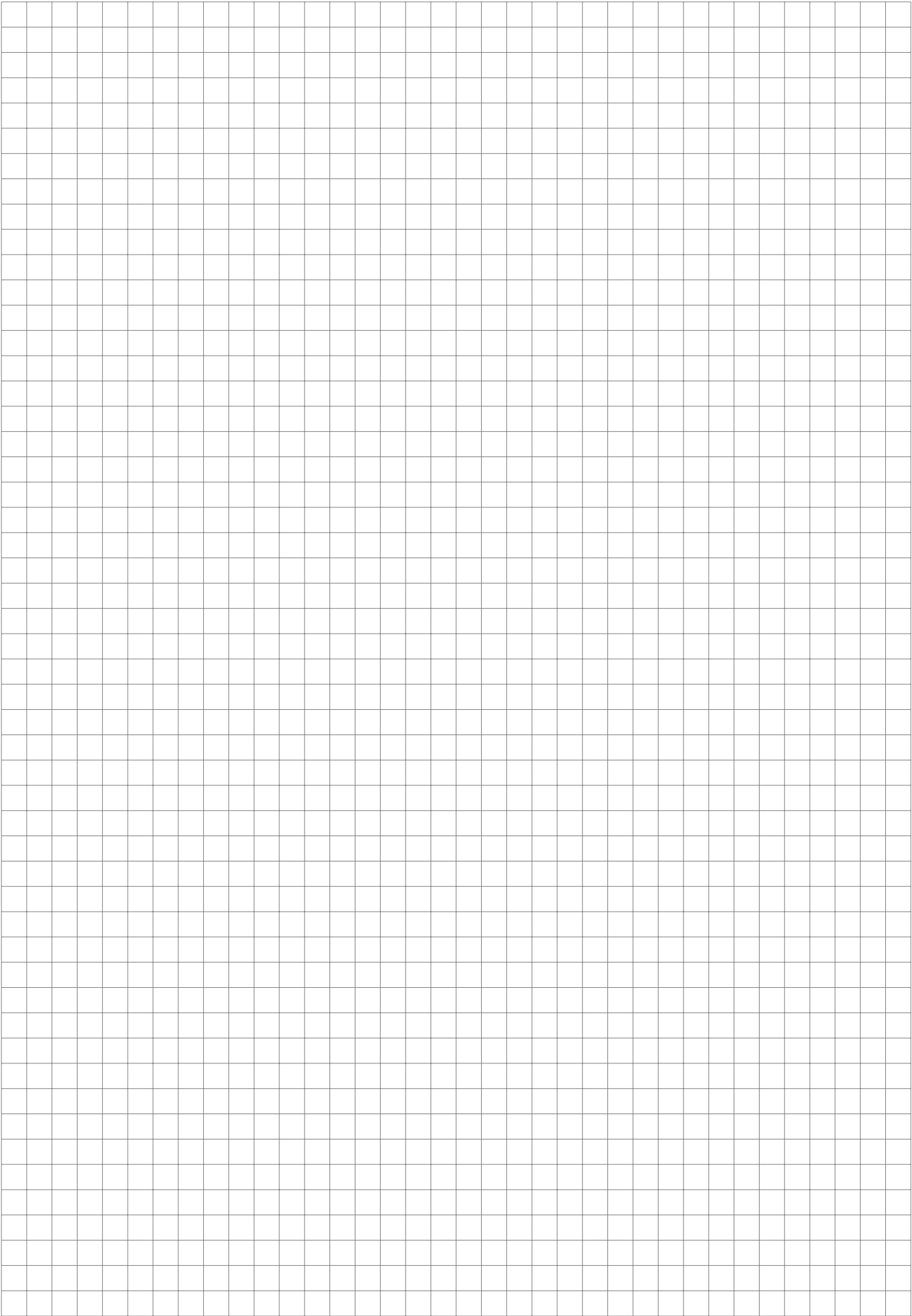
as $(A * B) \vee C \ltimes B \vee A \ltimes D$. The pasting law for homotopy pushouts can be used without proof.

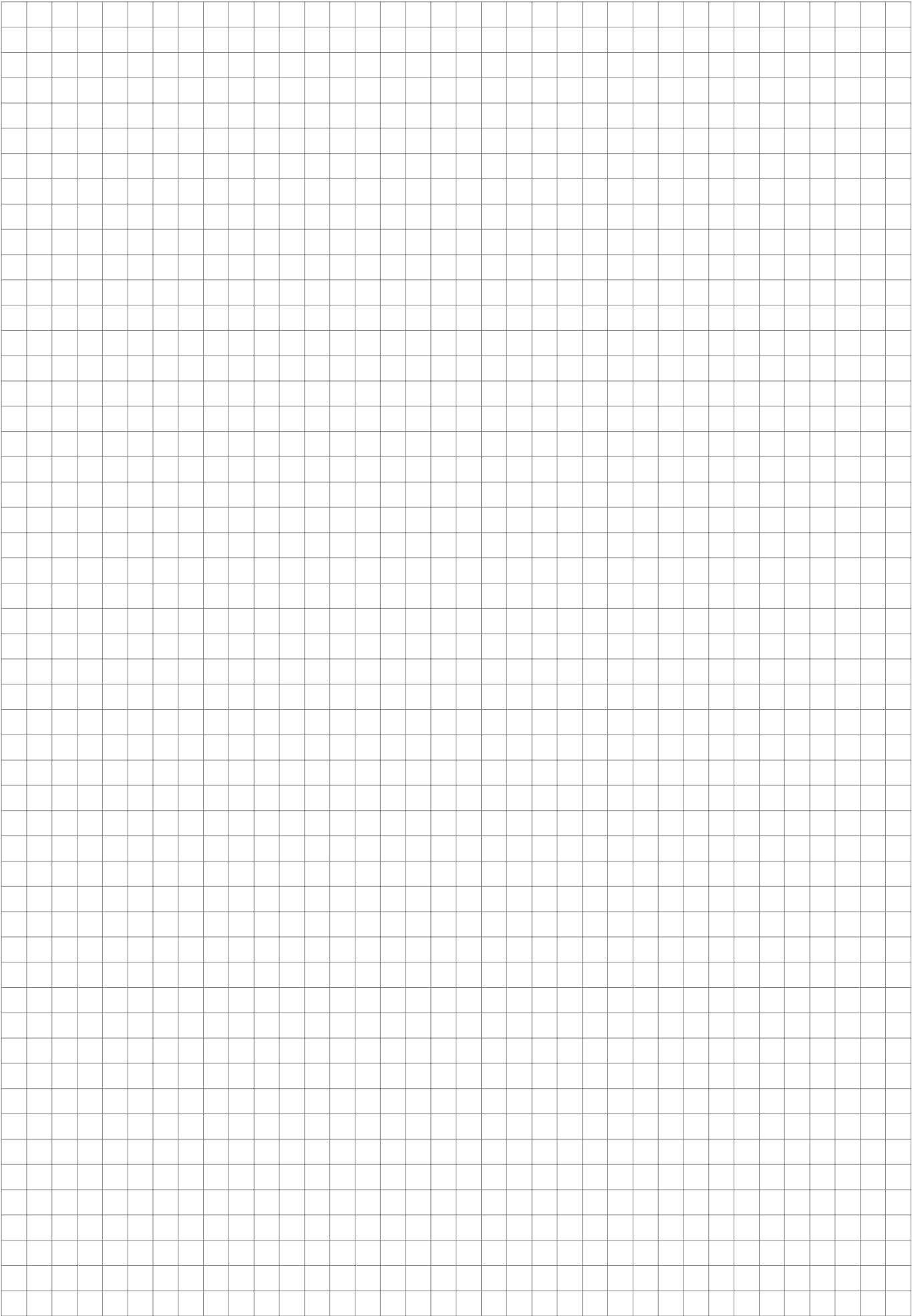
- (c) Write p_2 for the projection on the second factor of a product of two spaces and q_2 for the projection on the second factor of a wedge of two spaces. Identify the homotopy pushout of

$$B \vee C \xleftarrow{p_2} A \times (B \vee C) \xrightarrow{id_A \times q_2} A \times C$$

as $(A * B) \vee C$. Part (b) is not necessary to solve (c) and a Fubini argument for iterated homotopy pushout can be useful.







Question 4. An unpointed mapping space. (15 points) Let $(X; x_0)$ be a path connected pointed space, $(S^1, 1)$ be the unit circle in \mathbb{C} , and $\text{map}(S^1, X)$ be the space of all unpointed maps. We write $c_x: S^1 \rightarrow X$ for the constant map to x and $ev: \text{map}(S^1, X) \rightarrow X$ for the evaluation at 1, which we view as a pointed map by using c_{x_0} as a base point for $\text{map}(S^1, X)$.

- (a) Show that the map $x \mapsto c_x$ defines a pointed section of ev .
- (b) Show that $\pi_n(\text{map}(S^1, X); c_{x_0}) \cong \pi_n(X; x_0) \times \pi_{n+1}(X; x_0)$ for all $n \geq 2$.
- (c) Let $(X; x_0)$ be a path-connected H -space with a base point acting as a strict unit for the multiplication m . Construct a map $f: \Omega X \times X \rightarrow \text{map}(S^1, X)$ inducing an isomorphism on π_n for all $n \geq 1$ for the same base points as above.
- (d) Under the same assumptions as in (c), prove that f induces a bijection on π_0 . You can first prove that for any two loops $\omega, \alpha \in \Omega X$, there exists an unpointed homotopy $H: S^1 \times I \rightarrow X$ between ω and itself such that $H(1, -) = \alpha$.
- (e) Show that in general $\text{map}(S^1, X)$ is not weakly equivalent to the product $\Omega X \times X$. You can use $X = S^1 \vee S^1$ and find two distinct pointed homotopy classes $S^1 \rightarrow X$ which are homotopic as unpointed maps.



