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| Name      |  | First name |  |
| Signature |  | Sciper     |  |

Justifications and explanations based on the theory seen in class are required in each exercise.

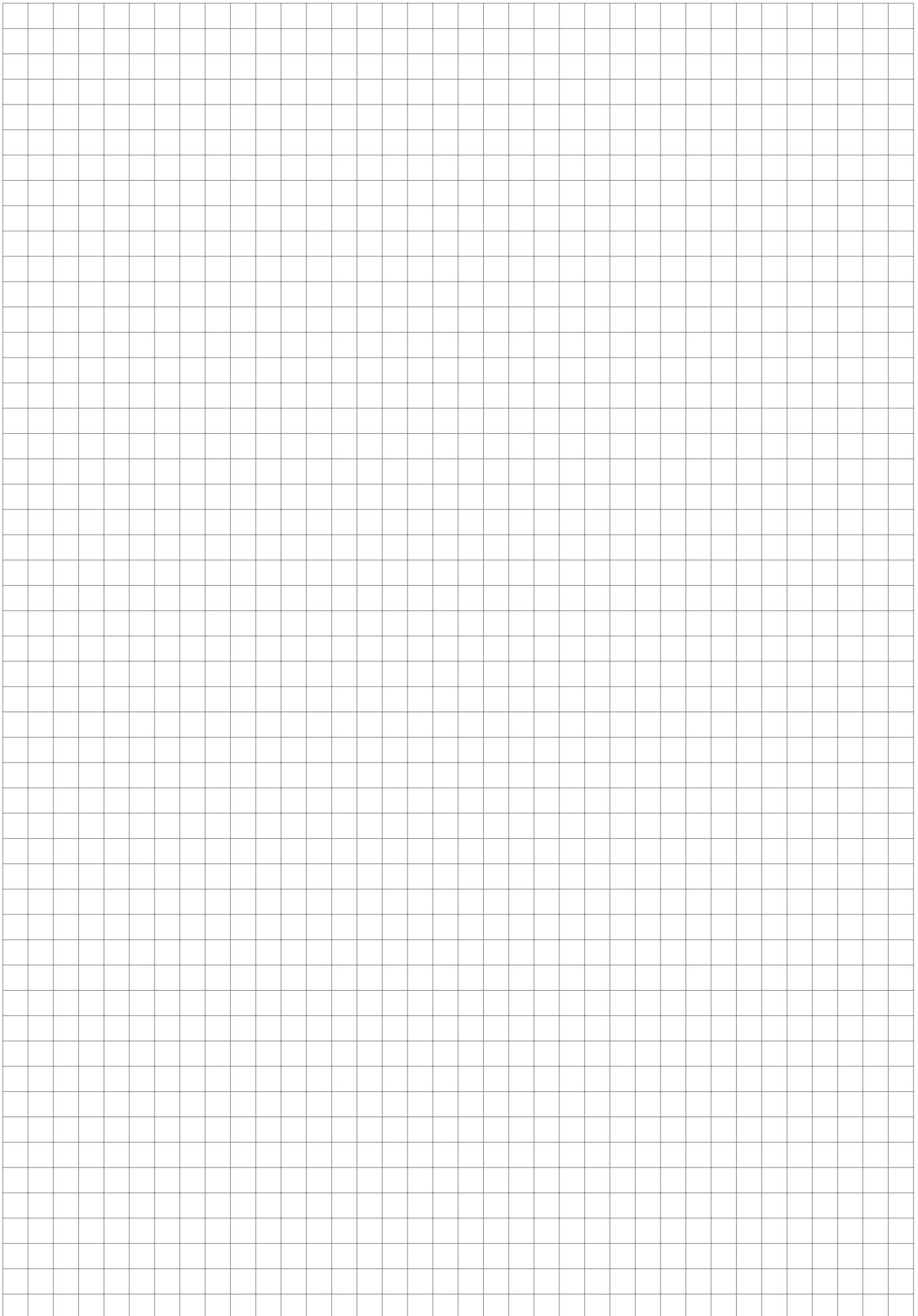
The following table is reserved for correction.

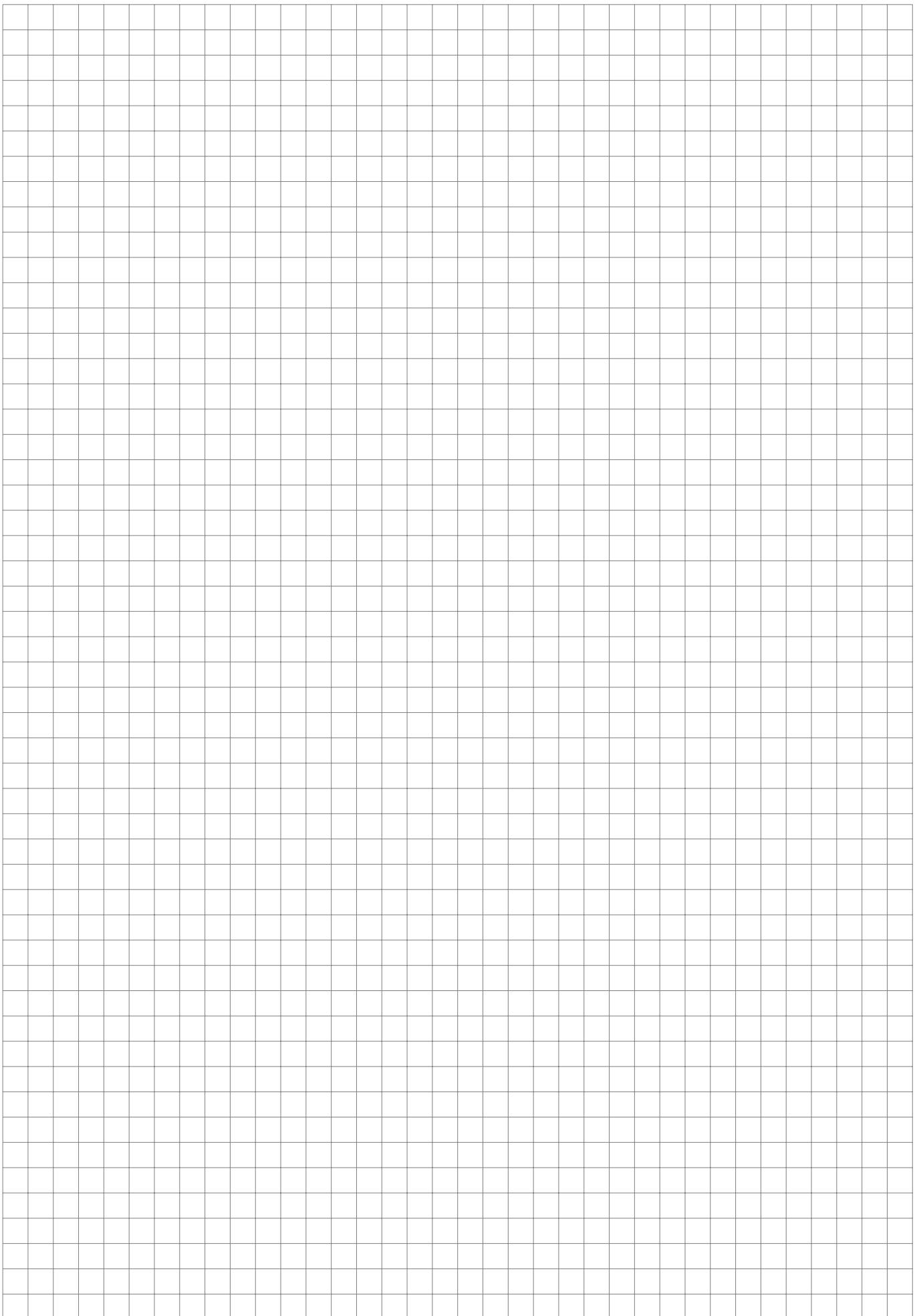
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|----------------|--|----------------|--|--------------------|--|
| Ex. 1 (15 pts) |  | Ex. 2 (15 pts) |  | Ex. 3 (15 pts)     |  |
| Ex. 4 (15 pts) |  | Written Ex.    |  |                    |  |
| Total (60 pts) |  | Oral Ex.       |  | <b>Final Grade</b> |  |

**Question 1, a computation.** (15 points) The final goal of this exercise is the computation of the set of pointed homotopy classes of maps  $[\mathbb{R}P^2, S^2]$ . We use the notation  $c$  for the constant map,  $p: S^2 \rightarrow \mathbb{R}P^2$  for the quotient by the antipodal action of  $C_2$ , and  $q: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2/\mathbb{R}P^1$  for the quotient map. Apply the following strategy.

- (a) Identify the first six terms of the Puppe sequence for the degree 2 map  $f: S^1 \rightarrow S^1$ .
- (b) Obtain a long exact sequence of sets of pointed homotopy classes of maps into  $S^2$ .
- (c) Perform then a careful analysis of exactness (sometimes of sets, not of groups !) so as to identify one representative  $\mathbb{R}P^2 \rightarrow S^2$  for each homotopy class, namely the images under  $q^*$  of any self-map of  $S^2$  of even, respectively odd, degree.
- (d) Show that there does not exist any map  $f: \mathbb{R}P^2 \rightarrow S^2$  such that  $f \circ p$  is homotopic to the identity. Find then a space  $X$  such that  $\pi_n X \cong \pi_n \mathbb{R}P^2$  for all  $n \geq 0$  but is not homotopy equivalent to  $\mathbb{R}P^2$ .

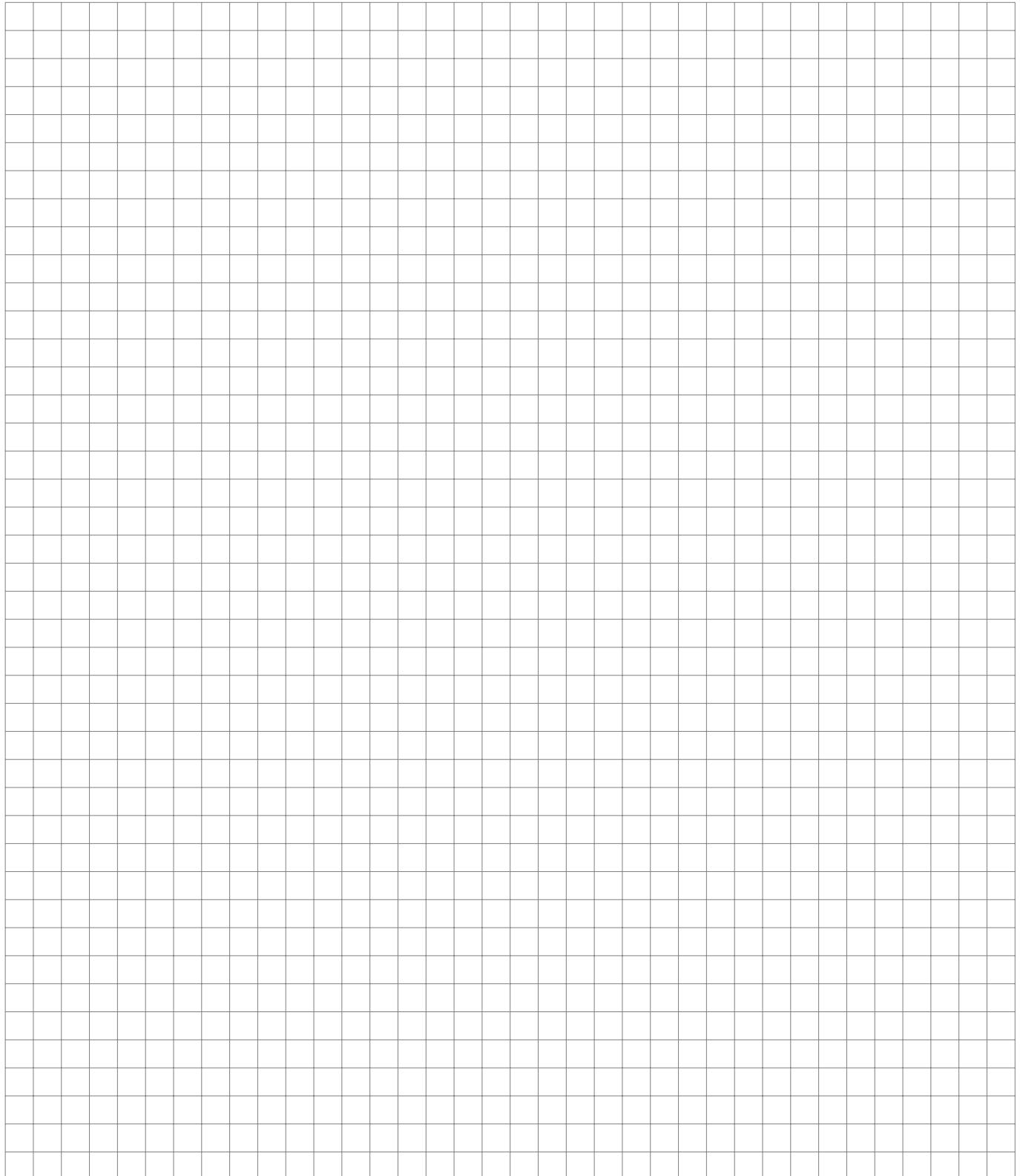
**Note.** All results about the real projective spaces, quotient maps from the Topology class, the CW-structure of real projective spaces from the Algebraic Topology class can be used. In particular you can use the fact that  $q \circ p$  has degree zero.

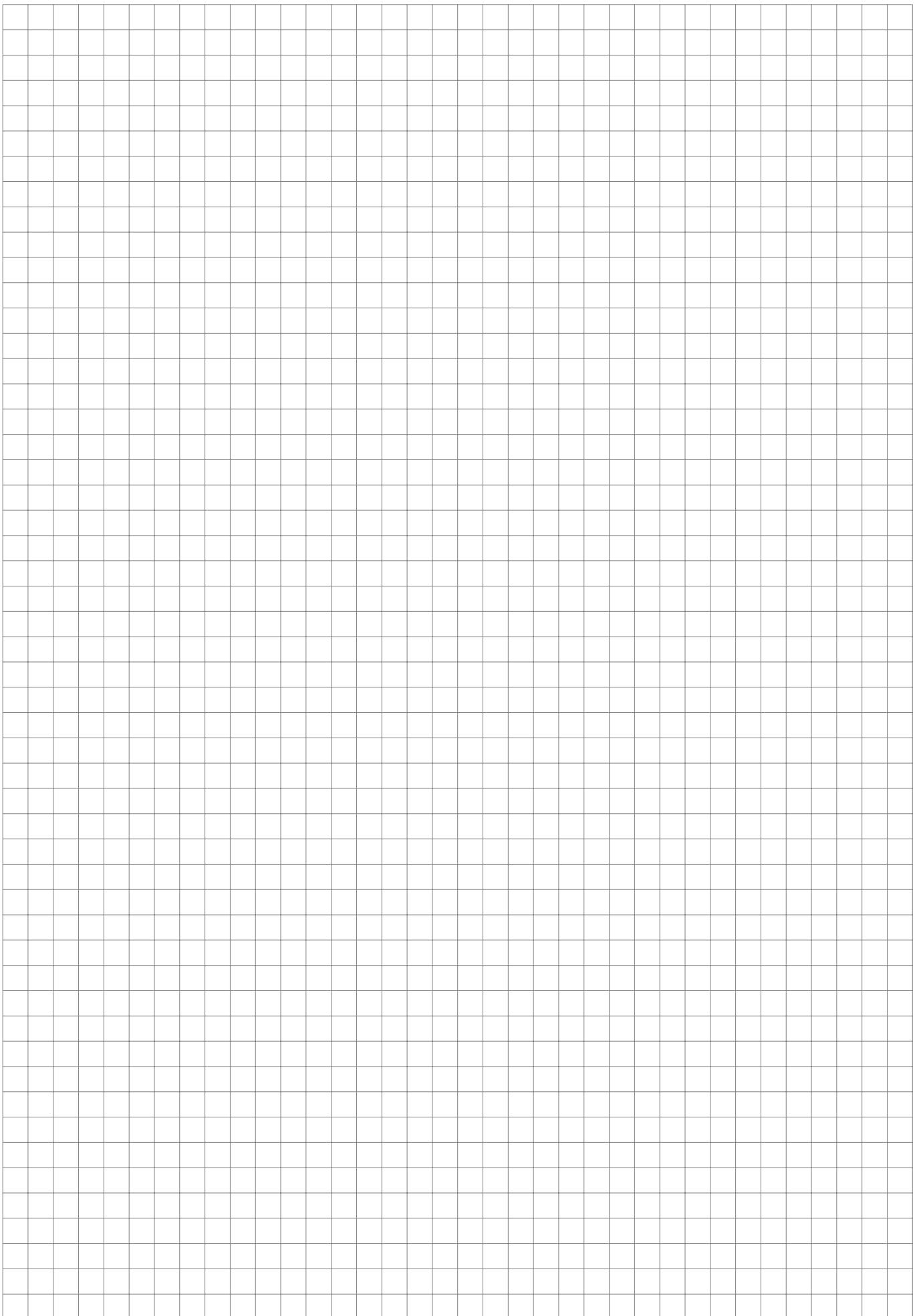


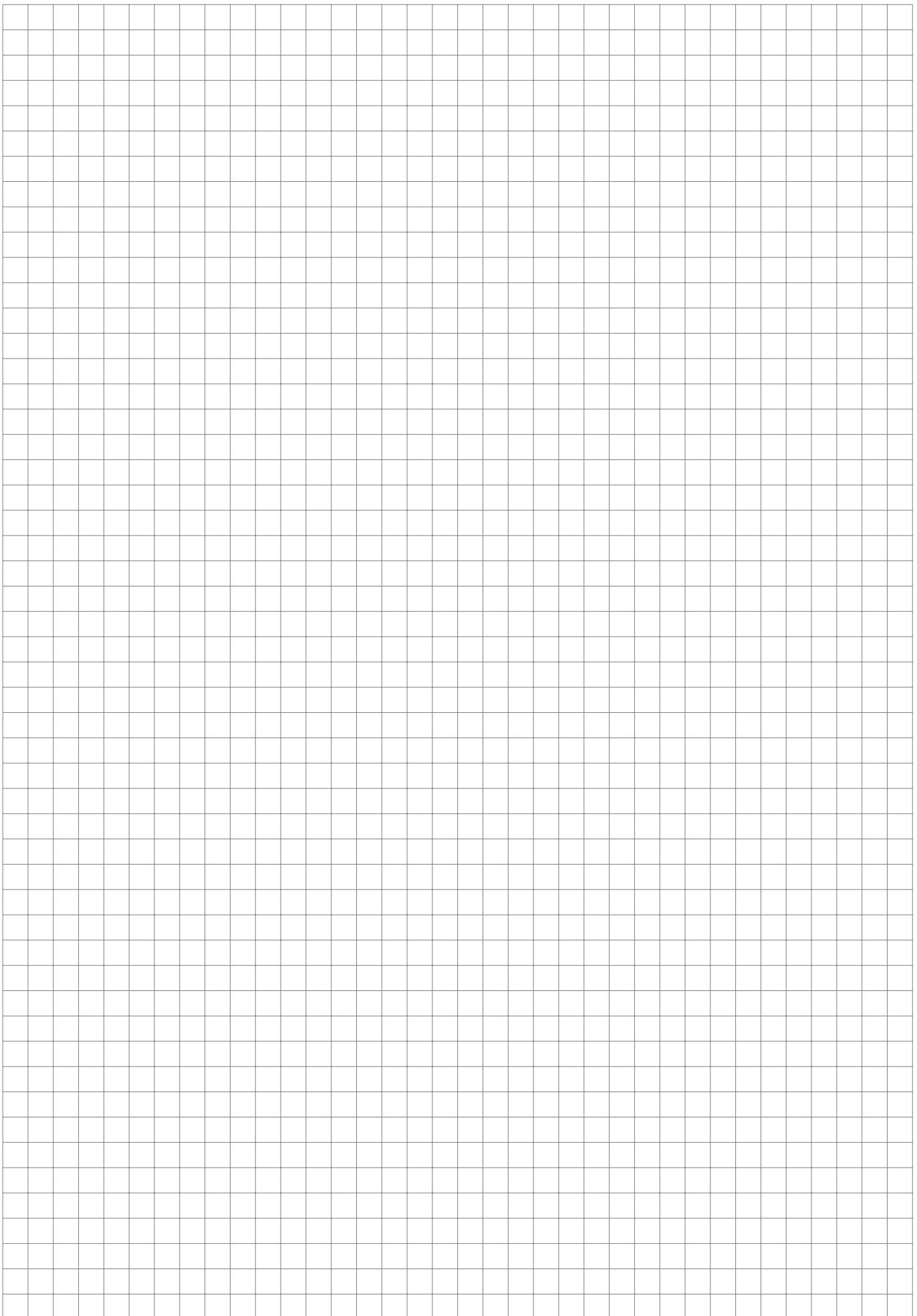


**Question 2, a complement to the Hurewicz Theorem.** (15 points) Let  $n \geq 2$  and  $X$  be a pointed  $(n-1)$ -connected space. The final goal of this exercise is to prove that the Hurewicz homomorphism  $Hu: \pi_{n+1}X \rightarrow H_{n+1}(X; \mathbb{Z})$  is surjective. You will first explain how to reduce the proof to the case of a CW-complex of dimension  $n+1$ . Show by an example that  $Hu$  is not an isomorphism in general.

**Hint.** To prove surjectivity of  $Hu$  for a CW-complex  $X$  of dimension  $n+1$  we suggest to fix a cycle  $\sigma \in C_{n+1}^{cell}(X)$  and represent a preimage under  $Hu$  as a pushout of a natural transformation of diagrams from  $D^{n+1} \leftarrow S^n \rightarrow D^{n+1}$  to the pushout diagram defining the CW-complex  $X$ .







**Question 3, Two homotopy pushouts.** (15 points) Let  $(A, a_0), (B, b_0), (C, c_0)$ , and  $(D, d_0)$  be well-pointed, locally compact, and Hausdorff spaces. Write  $c: A \rightarrow C$  and  $d: B \rightarrow D$  for the constant maps to the respective base points, and define the half-smash  $A \ltimes D = (A \times D)/(A \times d_0)$ .

(a) Explain why  $A \times (-)$  converts homotopy pushout squares into homotopy pushout squares.  
 (b) Identify the homotopy pushout of the diagram

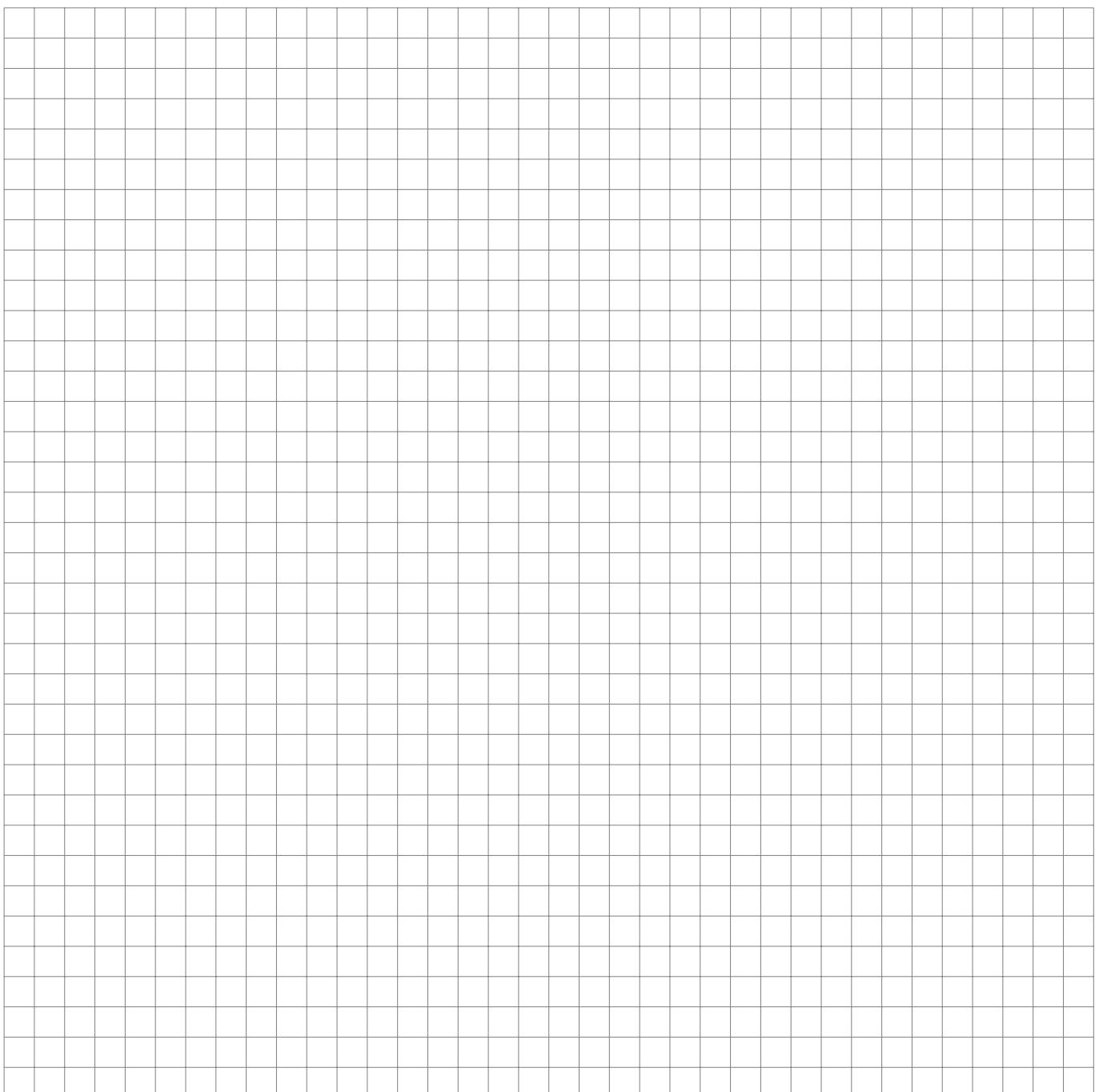
$$A \times D \xleftarrow{id_A \times d} A \times B \xrightarrow{c \times id_B} C \times B$$

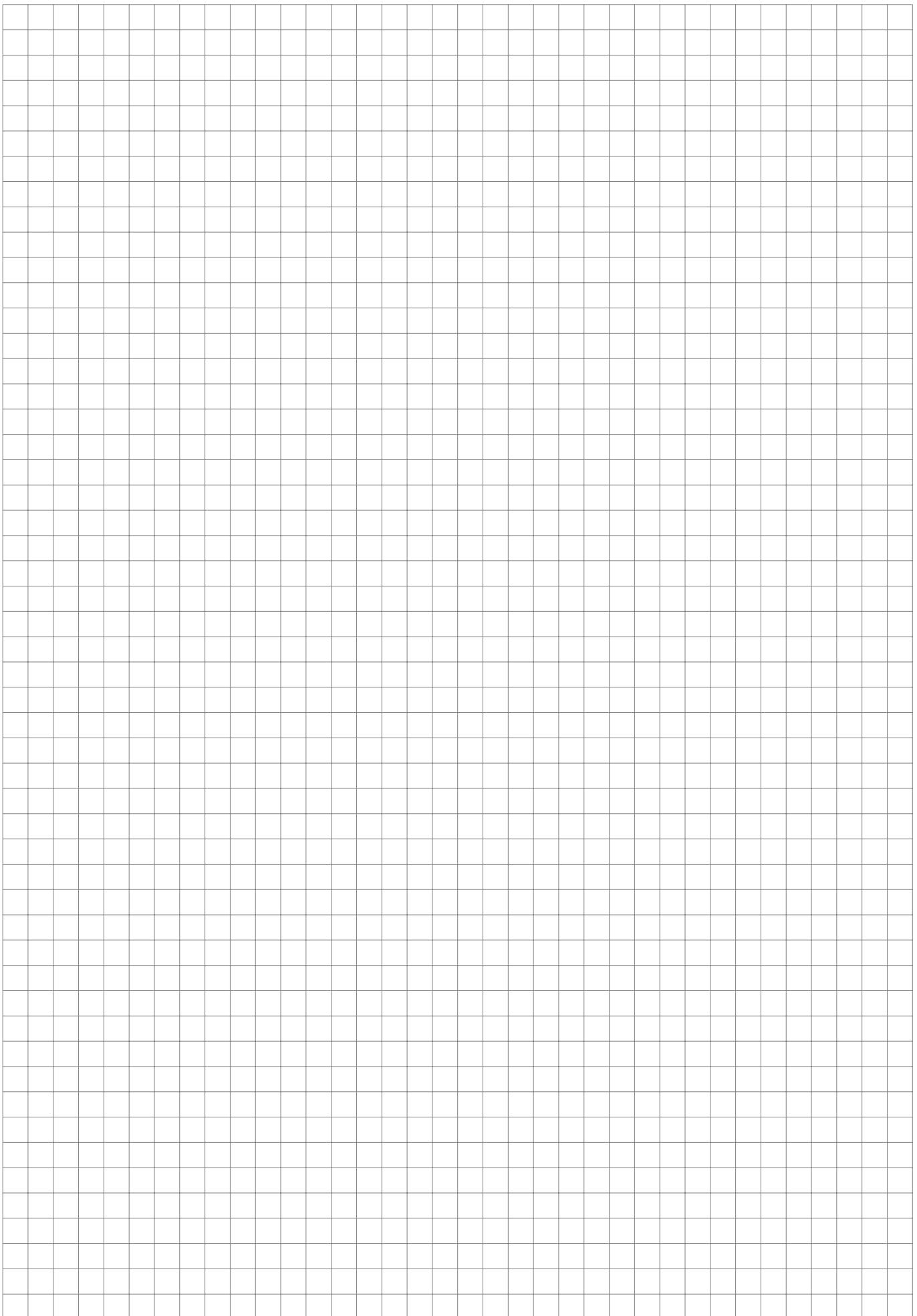
as  $(A * B) \vee C \times B \vee A \times D$ . The pasting law for homotopy pushouts can be used without proof.

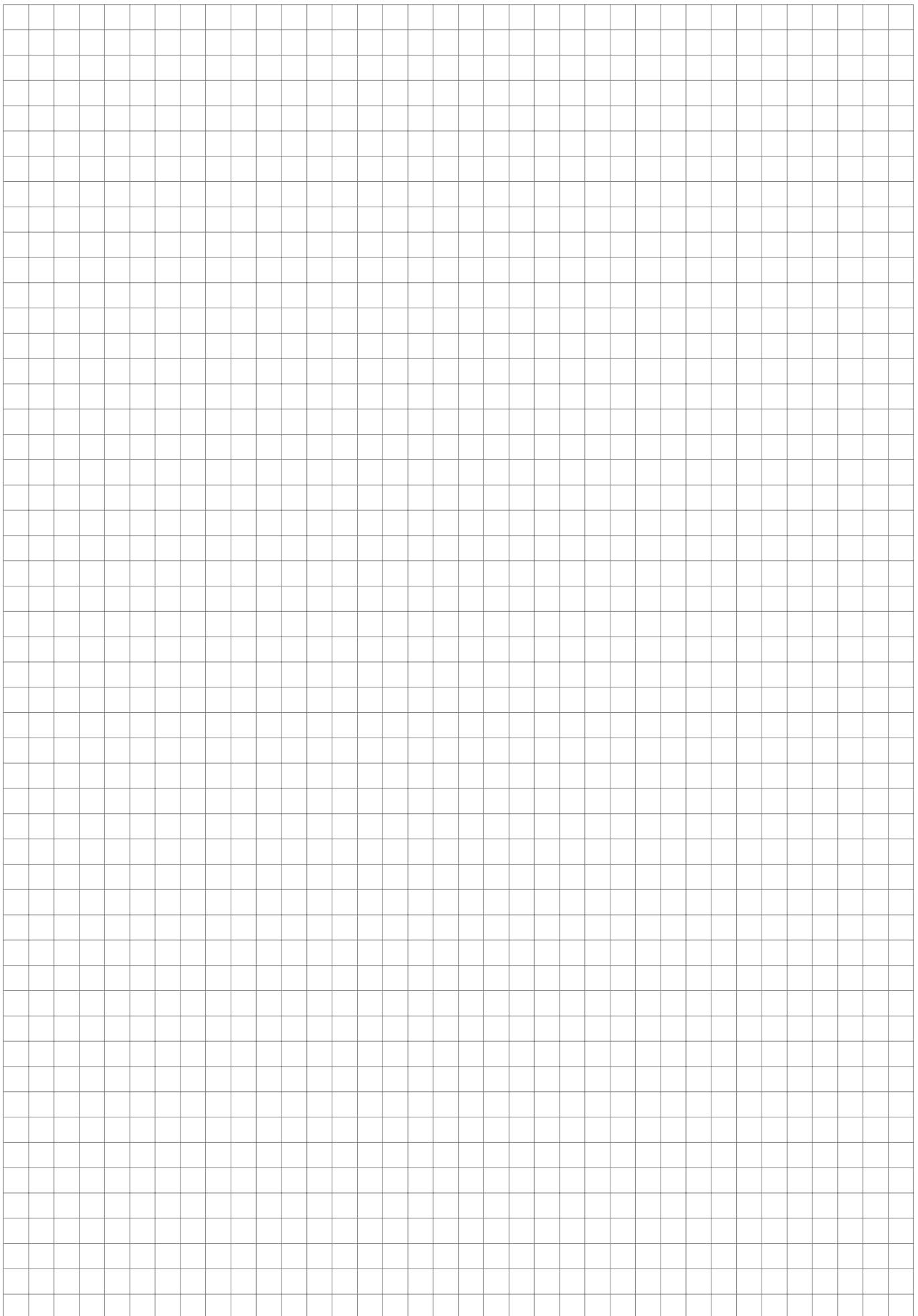
(c) Write  $p_2$  for the projection on the second factor of a product of two spaces and  $q_2$  for the projection on the second factor of a wedge of two spaces. Identify the homotopy pushout of

$$B \vee C \xleftarrow{p_2} A \times (B \vee C) \xrightarrow{id_A \times q_2} A \times C$$

as  $(A * B) \vee C$ . Part (b) is not necessary to solve (c) and a Fubini argument for iterated homotopy pushout can be useful.







**Question 4. An unpointed mapping space.** (15 points) Let  $(X; x_0)$  be a path connected pointed space,  $(S^1, 1)$  be the unit circle in  $\mathbb{C}$ , and  $map(S^1, X)$  be the space of all unpointed maps. We write  $c_x: S^1 \rightarrow X$  for the constant map to  $x$  and  $ev: map(S^1, X) \rightarrow X$  for the evaluation at 1, which we view as a pointed map by using  $c_{x_0}$  as a base point for  $map(S^1, X)$ .

(a) Show that the map  $x \mapsto c_x$  defines a pointed section of  $ev$ .

(b) Show that  $\pi_n(\text{map}(S^1, X); c_{x_0}) \cong \pi_n(X; x_0) \times \pi_{n+1}(X; x_0)$  for all  $n \geq 2$ .

(c) Let  $(X; x_0)$  be a path-connected  $H$ -space with a base point acting as a strict unit for the multiplication  $m$ . Construct a map  $f: \Omega X \times X \rightarrow \text{map}(S^1, X)$  inducing an isomorphism on  $\pi_n$  for all  $n \geq 1$  for the same base points as above.

(d) Under the same assumptions as in (c), prove that  $f$  induces a bijection on  $\pi_0$ . You can first prove that for any two loops  $\omega, \alpha \in \Omega X$ , there exists an unpointed homotopy  $H: S^1 \times I \rightarrow X$  between  $\omega$  and itself such that  $H(1, -) = \alpha$ .

(e) Show that in general  $\text{map}(S^1, X)$  is not weakly equivalent to the product  $\Omega X \times X$ . You can use  $X = S^1 \vee S^1$  and find two distinct pointed homotopy classes  $S^1 \rightarrow X$  which are homotopic as unpointed maps.



