EPFL - Fall Semester 2021-2022	Jérôme Scherer
Homotopy Theory	**Final Examen**
Mathematics	17 January 2022
Duration : 180 minutes	Number of points : 60

Name		First name	
Signature		Sciper	

Justifications and explanations based on the theory seen in class are required in each exercise.

The following table is reserved for correction.

Ex. 1 (15 pts)		Ex. 2 (15 pts)		Ex. 3 (15 pts)	
Ex. 4 (15 pts)		Written Ex.			
Total (60 pts)		Oral Ex.		**Final Grade**	

Question 1, a computation. (15 points) The final goal of this exercise is the computation of the set of pointed homotopy classes of maps $[\mathbb{R}P^2, S^2]$. We use the notation c for the constant map, $p\colon S^2 \to \mathbb{R}P^2$ for the quotient by the antipodal action of C_2, and $q\colon \mathbb{R}P^2 \to \mathbb{R}P^2/\mathbb{R}P^1$ for the quotient map. Apply the following strategy.

(a) Identify the first six terms of the Puppe sequence for the degree 2 map $f\colon S^1 \to S^1$.

(b) Obtain a long exact sequence of sets of pointed homotopy classes of maps into S^2.

(c) Perform then a careful analysis of exactness (sometimes of sets, not of groups!) so as to identify one representative $\mathbb{R}P^2 \to S^2$ for each homotopy class, namely the images under q^* of any self-map of S^2 of even, respectively odd, degree.

(d) Show that there does not exist any map $f\colon \mathbb{R}P^2 \to S^2$ such that $f \circ p$ is homotopic to the identity. Find then a space X such that $\pi_n X \cong \pi_n \mathbb{R}P^2$ for all $n \geq 0$ but is not homotopy equivalent to $\mathbb{R}P^2$.

Note. All results about the real projective spaces, quotient maps from the Topology class, the CW-structure of real projective spaces from the Algebraic Topology class can be used. In particular you can use the fact that $q \circ p$ has degree zero.

Question 2, a complement to the Hurewicz Theorem. (15 points) Let $n \geq 2$ and X be a pointed $(n-1)$-connected space. The final goal of this exercise is to prove that the Hurewicz homomorphism $Hu\colon \pi_{n+1}X \to H_{n+1}(X;\mathbb{Z})$ is surjective. You will first explain how to reduce the proof to the case of a CW-complex of dimension $n+1$. Show by an example that Hu is not an isomorphism in general.

Hint. To prove surjectivity of Hu for a CW-complex X of dimension $n+1$ we suggest to fix a cycle $\sigma \in C^{cell}_{n+1}(X)$ and represent a preimage under Hu as a pushout of a natural transformation of diagrams from $D^{n+1} \leftarrow S^n \to D^{n+1}$ to the pushout diagram defining the CW-complex X.

Question 3, Two homotopy pushouts. (15 points) Let $(A, a_0), (B, b_0), (C, c_0)$, and (D, d_0) be well-pointed, locally compact, and Hausdorff spaces. Write $c\colon A \to C$ and $d\colon B \to D$ for the constant maps to the respective base points, and define the half-smash $A \ltimes D = (A \times D)/(A \times d_0)$.

(a) Explain why $A \times (-)$ converts homotopy pushout squares into homotopy pushout squares.

(b) Identify the homotopy pushout of the diagram

$$A \times D \xleftarrow{id_A \times d} A \times B \xrightarrow{c \times id_B} C \times B$$

as $(A * B) \vee C \rtimes B \vee A \ltimes D$. The pasting law for homotopy pushouts can be used without proof.

(c) Write p_2 for the projection on the second factor of a product of two spaces and q_2 for the projection on the second factor of a wedge of two spaces. Identify the homotopy pushout of

$$B \vee C \xleftarrow{p_2} A \times (B \vee C) \xrightarrow{id_A \times q_2} A \times C$$

as $(A * B) \vee C$. Part (b) is not necessary to solve (c) and a Fubini argument for iterated homotopy pushout can be useful.

Question 4. An unpointed mapping space. (15 points) Let $(X; x_0)$ be a path connected pointed space, $(S^1, 1)$ be the unit circle in $\mathbb{C}$, and $map(S^1, X)$ be the space of all unpointed maps. We write $c_x \colon S^1 \to X$ for the constant map to x and $ev \colon map(S^1, X) \to X$ for the evaluation at 1, which we view as a pointed map by using c_{x_0} as a base point for $map(S^1, X)$.

(a) Show that the map $x \mapsto c_x$ defines a pointed section of ev.

(b) Show that $\pi_n\left(map(S^1, X); c_{x_0}\right) \cong \pi_n(X; x_0) \times \pi_{n+1}(X; x_0)$ for all $n \geq 2$.

(c) Let $(X; x_0)$ be a path-connected H-space with a base point acting as a strict unit for the multiplication m. Construct a map $f \colon \Omega X \times X \to map(S^1, X)$ inducing an isomorphism on π_n for all $n \geq 1$ for the same base points as above.

(d) Under the same assumptions as in (c), prove that f induces a bijection on π_0. You can first prove that for any two loops $\omega, \alpha \in \Omega X$, there exists an unpointed homotopy $H \colon S^1 \times I \to X$ between ω and itself such that $H(1, -) = \alpha$.

(e) Show that in general $map(S^1, X)$ is not weakly equivalent to the product $\Omega X \times X$. You can use $X = S^1 \vee S^1$ and find two distinct pointed homotopy classes $S^1 \to X$ which are homotopic as unpointed maps.