Homotopy Theory

Jérôme Scherer

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Introduction

This is a Master's course taught for the first time in the Fall semester 2021. The following sequence of courses taught at the Bachelor's level constitute the natural background for the ideas we study here. In "Metric and Topological Spaces" the notion of topological space is introduced as a generalization of that of a metric space, the fundamental group is defined and computed for the circle. In my own course "Topology" the whole semester is spent around this important homotopy invariant and two different aspects are highlighted. The Seifert-van Kampen Theorem allows us to compute fundamental groups of spaces constructed by assembling elementary pieces, in particular in the case of quotient spaces, and the theory of coverings gives a more geometrical meaning to the fundamental group by identifying its elements as deck transformations. Finally, the course "Algebraic Topology" makes use of homological algebra and introduces homology groups as a new homotopy invariant, both in the form of singular homology and that of cellular homology, a version better suited for computations when the spaces one works with are CW-complexes.

In this course we focus mainly on higher homotopy groups. Several excellent textbooks serve as inspiration for this course. We do not claim any originality and rely often on the approach and technical tools presented in May, etc.

CHAPTER 1

Topological complements

In this short chapter we cover a subject which could have been part of the topology course, but as it has not been done there, we have to do it now. We know that (topological) spaces and (continuous) maps form a category, and we will see that its structure is even richer as the *set* of morphisms between two spaces comes equipped with a nice topology, so we have in fact a *space* of maps. This topology, called compact-open for obvious reasons, is designed so that several desirable properties hold, such as the exponential law. These properties hold unfortunately only for a restricted class of spaces and we will focus on them in this chapter. In the last section we indicate how to deal more seriously with this issue, but will not give full proofs.

1. Compact-open topology

We follow Hatcher's book [**3**, Appendix A]. Our aim is to introduce and study a (good) topology for the set of all maps. To distinguish this mapping space from the set of all maps we will use a different notation. The set $\mathrm{mor}(X, Y)$ of all maps will thus become a space $\mathrm{map}(X, Y)$. We will also use the notation Y^X for the set of all non necessarily continuous maps $X \to Y$.

Definition 1.1. Let X, Y be two spaces, $K \subset X$ be compact and $U \subset Y$ open. We define $B(K, U) = \{f : X \to Y \mid f(K) \subset U\}$. The *compact-open topology* is defined on the set of all maps $f : X \to Y$ by the subbasis $B(K, U)$ where K runs over all compact subspaces of X and U over all open subspaces of Y. The space $\mathrm{map}(X, Y)$ of all maps is called *mapping space*.

Therefore a basis for the compact-open topology is given by finite intersections of $B(K, U)$'s.

Definition 1.2. Let X, Y be two spaces, $n \geq 1$ an integer, K_i be compact subspaces of X and U_i be open subspaces of Y for all $1 \leq i \leq n$. We define $B(K_\bullet, U_\bullet) = \{f\colon X \to Y \mid f(K_i) \subset U_i \text{ for all } 1 \leq i \leq n\}$ for a basis of the compact-open topology.

Finally recall that an arbitrary open subset of $\mathrm{map}(X, Y)$ is an arbitrary union of basic open subsets. Many important examples will have a compact source.

Example 1.3. When $X = *$ is a singleton, the evaluation at this unique point provides a homeomorphism $\mathrm{map}(*, Y) \approx Y$. More generally, when $X = [n] = \{1, \dots, n\}$ is a finite discrete space, we have a homeomorphism $\mathrm{map}([n], Y) \approx Y^n$.

Example 1.4. When $X = I = [0, 1]$ equipped with the subspce topology of the metric space $\mathbb{R}$, one can choose a simpler basis for the compact-open topology on $\mathrm{map}(I, Y)$, the space of all paths in Y. For any choice of $0 \leq t_0 < t_1 < \cdots < t_{n-1} < t_n \leq 1$ and open subsets $U_1, \dots, U_n$ in Y we ask that $f([t_{i-1}, t_i]) \subset U_i$.

Example 1.5. When $X = S^1$ we call $\Lambda Y = \mathrm{map}(S^1, Y)$ the space of free loops in Y since an element in λY is a loop $\lambda\colon S^1 \to Y$ starting at any point $y = \lambda(1)$.

2. The exponential law

One of the main features for the compact-open topology is an exponential law. Let us recall here the set theoretic version. Let X, Y, Z be sets. Then $Z^{X\times Y} \cong (Z^Y)^X$. We will use the explicit form of this bijection:

$$f\colon X \times Y \to Z \text{ corresponds to } \varphi = a(f)\colon X \to Z^Y, x \mapsto f(x, -)$$

This isomorphism is natural in Y, so we have an *adjunction* $(X \times -) \dashv Z^{(-)}$. This explains the name $a(f)$ for the adjoint of the morphism f.

Now we move to the topological version of this adjunction. The set of maps $\mathrm{map}(X \times Y, Z)$ is a subset of $Z^{X\times Y}$ and likewise $(Z^Y)^X$ contains $\mathrm{map}(X, \mathrm{map}(Y, Z))$. In order to control the behavior of the adjoint map, we will need to assume that one space is locally compact, i.e., every neighborhood of a given point contains a compact neighborhood.

Lemma 2.1. *Let X be a locally compact space. Then the evaluation map*

$$\mathrm{ev}\colon \mathrm{map}(X,Y)\times X\to Y,$$

defined by $\mathrm{ev}(f,x)=f(x)$*, is continuous.*

Proof. Let V be an open subset in Y and consider a pair (f,x) in the domain such that $\mathrm{ev}(f,x)\in V$. Since f is continuous, the preimage $f^{-1}(V)$ is open in X and by assumption there exists a compact neighborhood $x\in K\subset f^{-1}(V)$. Thus $f(K)\subset V$. We choose therefore the basic open subset $B(K,V)$ in the mapping space and K as a neighborhood of x in X.

Then $\mathrm{ev}(g,k)=g(k)\in V$ for all $g\in B(K,V)$ and $k\in K$, which shows that the evaluation map sends the neighborhood $B(K,V)\times K$ of (f,x) to V. □

Proposition 2.2. *Let Y be a locally compact space. Then a map $f\colon X\times Y\to Z$ is continuous if and only if its adjoint $a(f)=\varphi\colon X\to \mathrm{map}(Y,Z)$ is so.*

Proof. Notice first that the restriction $f\,|_{x\times Y}$ of a continuous map f is again continuous for any $x\in X$. This justifies the fact that the map $a(f)$ belongs to the set of set theoretic morphisms $X\to\mathrm{map}(Y,Z)$ and not only to $(Z^Y)^X$.

Assuming that f is continuous, we have to show that so is $a(f)$. Given $L\subset Y$ compact and $W\subset Z$ open, let us check that $a(f)^{-1}(B(L,W))$ is open in X. This inverse image consists of those $x\in X$ such that $f(x\times L)\subset W$. Sor for each such x we have to find an open neighborhood U such that $f(U)\subset B(L,W)$.

Now, since W is open, so is $f^{-1}(W)$ and it contains $x\times L$. By definition of the product topology there exists inside $f^{-1}(W)$ a union of open boxes $U_i\times V_i$ containing $x\times L$. Because L is compact, a finite number of such boxes suffices and we can choose $U=\cap U_i$ as an open subset in X, and $V=\cup V_i$.

What we have achieved is that $x\times L\subset U\times V\subset f^{-1}(W)$. Therefore U is an open neighborhood of x and $a(f)(U)\subset B(L,W)$.

To prove the other implication, assume now that $a(f)$ is continuous and consider the composite

$$X\times Y\xrightarrow{a(f)\times Y}\mathrm{map}(Y,Z)\times Y\xrightarrow{\mathrm{ev}} Z$$

This is a continuous map as the evaluation map is continuous by Lemma 2.1 and one checks that it coincides with f. Notice that we have used here the assumption on Y to be locally compact. □

In order to establish the topological exponential law, we will pick better suited subbases for the mapping spaces coming into play. We start with the following lemma where no locally compactness needs to be assumed.

LEMMA 2.3. *Let X and Y be Hausdorff spaces and Z be any space. Then $\mathrm{map}(X \times Y, Z)$ admits as a subbasis for the compact-open topology all $B(K \times L, W)$ for K compact in X, L compact in Y, and W open in Z.*

PROOF. In principle we should use all compact subsets in $A \subset X \times Y$, not only boxes $K \times L$ as we claim here. So let us consider $f \in B(A, W)$ for an arbitrary compact subset A. We will prove that this subbasic open subset can also be described as an open subset for the topology generated by the more restrictive choice.

Since A is compact, so are its projection A_X and A_Y on X, respectively on Y. However if $f(A) \subset W$, it is not true in general that the image of this larger box, $f(A_X \times A_Y)$ is also contained in W. For any $(x, y) \in A$ we have $f(x, y) \in W$. Let us choose open neighborhoods $x \in U_x$ in A_X and $y \in U_y$ in A_Y such that $f(U_x \times V_y) \subset W$ such that $f(U_x \times V_x) \subset W$.

By assumption the space A_X is Hausdorff, and compact, thus normal, and we can separate x from the closed subset $A_X \setminus U_x$ by an open $x \in U$ such that $\overline{U} \subset U_x$. Likewise $y \in V$ with $V \subset V_y$. The compact subspaces $\overline{U} \times \overline{V}$ cover A, we extract now a finite cover $\overline{U_i} \times \overline{V_i}$. We have done that so as to make sure $f(\overline{U_i} \times \overline{V_i}) \subset W$, thus $f \in \cap B(\overline{U_i} \times \overline{V_i}, W)$. This shows that any $f \in B(A, W)$ admits a (basic) open neighborhood for the topology described by the choices in the lemma. □

Now that we have a more suitable description of the compact-open topology on $\mathrm{map}(X \times Y, Z)$, we observe that the set theoretical exponential sends $B(K \times L, W)$ to $B(K, B(L, W))$ by Proposition 2.2. We show next that such special subbasic subsets also form a subbasis for the compact-open topology on $\mathrm{map}(X, \mathrm{map}(Y, Z))$.

LEMMA 2.4. *Let X be a Hausdorff space and $\mathcal{B}$ a subbasis for the topology of a space Y. Then $B(K, \mathcal{B})$, for compact $K \subset X$ form a subbasis of the compact-open topology on* $\mathrm{map}(X, Y)$.

PROOF. Similarly to the previous proof of Lemma 2.3 we consider $f \in B(K, V)$, a subbasic open subset in $\mathrm{map}(X, Y)$ and will exhibit a finite number of compact subset $K_i \subset X$ and subbasic $B_i \in \mathcal{B}$ such that $f \in \cap B(K_i, B_i) \subset B(K, V)$.

Let us write V as a union of basic subsets V_α, i.e. each V_α is a finite intersection $\cap V_{\alpha,j}$ for some $V_{\alpha,j} \in \mathcal{B}$. Since $f(K) \subset V$, the $f^{-1}(V_\alpha)$'s cover K, and we immediately extract a finite cover $V_1, \dots, V_m$. We can choose (as we will see in an exercise) compact subsets $K_i \subset f^{-1}(V_i)$ such that $K = \cup K_i$.

We have done this so that $f(K_i) \subset V_i$, i.e. $f \in B(K_i, V_i)$ for all $1 \leq i \leq m$. We conclude by expressing each

$$B(K_i, V_i) = B(K_i, \cap V_{i,j}) = \cap B(K_i, V_{i,j})$$

This means precisely that $f \in \bigcap_{i,j} B(K_i, V_{i,j})$, a finite intersection of the desired form, contained in $B(K, V)$. □

We are finally ready for our main result. We will go through the steps hinted at in the previous preparatory lemmas.

THEOREM 2.5. *Let X, Y be Hausdorff spaces and Y locally compact. We have a natural homeomorphism* $\mathrm{map}(X \times Y, Z) \approx \mathrm{map}(X, \mathrm{map}(Y, Z))$ *for any space Z.*

PROOF. We have seen in Proposition 2.2 that a map $f \colon X \times Y \to Z$ corresponds to a continuous adjoint $a(f)$ and vice-versa because Y is locally compact. This provides a bijection between these mapping spaces. It is a homeomorphism since the topology on $\mathrm{map}(X \times Y, Z)$ is defined by the subbasis $B(K \times L, W)$ (we use here that X and Y are Hausdorff and apply Lemma 2.3), and the operation a sends them to $B(K, B(L, W))$ which form a subbasis of $\mathrm{map}(X, \mathrm{map}(Y, Z))$ by Lemma 2.4 (here we need X Hausdorff). □

REMARK 2.6. What the exponential law tells us is first is that the functors $- \times Y$ and $\mathrm{map}(Y, -)$ form a pair of adjoint functors from topological spaces to topological spaces. Here we use the compact-open topology on $\mathrm{map}(Y, Z)$, but only remark that we have a natural bijection of sets:

$$\mathrm{mor}(X \times Y, Z) \cong \mathrm{mor}(X, \mathrm{map}(Y, Z))$$

But the exponential law tells us even more since these sets of morphisms can be given a topology in a way that the corresponding spaces of maps become homeomorphic. This yields a so-called *enriched adjunction*.

3. Some remarks on possible improvements

The assumption that all spaces are Hausdorff is a mild one. It is quite common to work with Hausdorff spaces, say in any course about metric spaces, or in algebraic topology when dealing with CW-complexes.

The second assumption we have made to prove the exponential law is not so harmless. Finite CW-complexes are compact, but in general arbitrary CW-complexes are not, and even worse, not locally compact. In fact a CW-complex is locally compact if and only if every open cell meets only finitely many closed cells. Hence, infinite dimensional projective spaces, infinite wedges of spheres are useful spaces that do not verify the assumptions made in the previous section.

We follow May's [**4**, Chapter 5] to indicate briefly a possible fix. In principle since, as we will see later, every space is "weakly" equivalent to a CW-complex, we would be happy to work with CW-complexes only. Two problems occur at least. The first one is that CW-complexes do not form a (co)complete category, sometimes (co)limits of CW-complexes give spaces which are not CW-complexes. The second one is related to the description of mapping spaces and the exponential law we have studied in the previous sections. Milnor as one of the leading mathematicians who suggested to work with CW-complexes. He proved that $\mathrm{map}(X, Y)$ is homotopy equivaéent to a CW-complex if X and Y are CW-complexes and X *is compact*. If not, there are examples where the resulting mapping space fails to be equivalent to a CW-complex. This category of spaces is hence not Cartesian closed.

Nowadays there are different ways to fix this problem. The solution I will describe in these notes consists in working with a nice subcategory of topological spaces, called compactly generated spaces. Another way around is to work in a different category, namely that of simplicial sets, a more combinatorial version of spaces which can be shown to have the same homotopy theory as spaces, in the sense of Quillen's theory of model categories, [**7**]. This is the topic of Kathryn's Hess course on abstract homotopy theory.

Definition 3.1. A space X is *weak Hausdorff* if any map $g\colon K \to X$ from a compact space K has a closed image $g(K) \subset X$.

If X is Hausdorff, then the image $g(K)$ being compact, it must be closed. This shows that Hausdorff is a stronger property than weak Hausdorff.

Definition 3.2. A subspace $A \subset X$ is *compactly closed* if, for any map $g\colon K \to X$ from a compact space K, the preimage $g^{-1}(A)$ is closed.

We give a characterization, without proof, of compactly closed subspaces in weak Hausdorff spaces.

Proposition 3.3. *If X is weak Hausdorff, $A \subset X$ is compactly closed if and only if the intersection $A \cap L$ is closed in X for any compact $L \subset X$.*

We finally arrive to the notion of k-space, where the letter k stands for 'kompakt" in German.

Definition 3.4. A space X is a *k-space* if every compactly closed subspace is closed in X. It is *compactly generated* if it is a weak Hausdorff k-space.

Hurewicz studied this kind of spaces first, in the 1930's, but it was much later, I believe in 1960's, that people realized formally the Cartesian closed property, see [**6**]. One reason why I don't wish to deal with this is that one has to change products into $k(X \times Y)$ the k-ification of the product, where all compactly closed subspaces are closed. Likewise the mapping space with its compact-open tiopology has to be replaced by its k-ification .

4. Pointed mapping spaces

It would be cleaner and easier to work with simplicial sets or compactly generated spaces, but as we did not develop the theory, we will content ourselves to work in a more restricted setting, or admit when really needed that the solution described in the previous section provides in particular an exponential law that holds in full generality.

In this section we work with pointed spaces (X, x_0), (Y, y_0), etc., and it is natural to consider thus pointed (based) maps.

DEFINITION 4.1. Let (X, x_0) and (Y, y_0) be pointed spaces. The space of *pointed maps* or *pointed mapping space* $\mathrm{map}_*((X, x_0), (Y, y_0))$ or $\mathrm{map}_*(X, Y)$ for short is the subspace of $\mathrm{map}(X, Y)$ consisting of all maps $f: X \to Y$ such that $f(x_0) = y_0$.

The space of pointed maps $\mathrm{map}_*(X, Y)$ is often considered as a pointed space itself, where we choose the constant map c_{y_0} as base point. Recall that the constant map c_{y_0} sends every point $x \in X$ to the base point $y_0 \in Y$.

We will develop now the theory for locally compact Hausdorff spaces, starting with some preliminaries. All results admit a generalization to compactly generated spaces, as is done for example in Strom's book [**11**, Chapter 3].

LEMMA 4.2. *Let Y be a Hausdorff space and consider the left hand side pushout square of locally compact Hausdorff spaces*

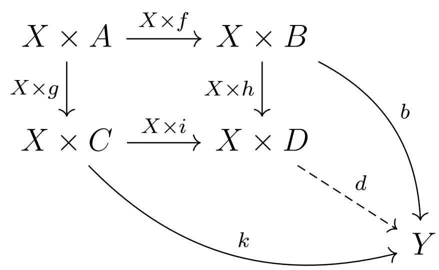
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{i} & D \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{map}(D, Y) & \xrightarrow{i^*} & \mathrm{map}(C, Y) \\ \downarrow{\scriptstyle h^*} & & \downarrow{\scriptstyle g^*} \\ \mathrm{map}(B, Y) & \xrightarrow{f^*} & \mathrm{map}(A, Y) \end{array}$$

Then the right hand side square is a pullback square.

PROOF. To verify that the square of mapping spaces is a pullback square we simply check that it enjoys the universal property. Given maps $\kappa: X \to \mathrm{map}(C, Y)$, adjoint to $k: X \times C \to Y$, and $\beta: X \to \mathrm{map}(B, Y)$, adjoint to $b: X \times B \to Y$, such that $g^* \circ \kappa = f^* \circ \beta$ we have to prove that there is a unique map $\delta: X \to \mathrm{map}(D, Y)$ making the appropriate diagram commutative.

The composite map $b \circ (X \times f)$ is adjoint to $f^* \circ \beta$, and likewise for $k \circ (X \times g)$ and $g^* \circ \kappa$. Since two different adjunctions come into play here, say for the first claim, whose left adjoint are respectively $- \times A$ and $- \times B$, we need to use the fact that they are *conjugate* via $- \times f$ and $f^* : \mathrm{map}(B, -) \to \mathrm{map}(A, -)$. More details can be found for example in Mac Lane's [**5**, Theorem IV.7.2]

This means that the solid arrow diagram below commutes:



Since $X \times -$ is a left adjoint, it preserves pushouts. This means that the above square is a pushout square, hence the dashed arrow d exists and is unique. Its adjoint $\delta\colon X \to \mathrm{map}(D, Y)$ solves the problem. □

This applies in particular to quotients and helps us understand mapping spaces out of such quotient spaces.

Corollary 4.3. *Let (A, a_0) be a subspace of a locally compact Hausdorff space (X, a_0), and (Y, y_0) be another locally compact Hausdorff space. Then* $\mathrm{map}_*(X/A, Y)$ *is homeomorphic to the subspace of* $\mathrm{map}(X, Y)$ *of all maps $f\colon X \to Y$ such that $f|_A = c_{y_0}$.*

Proof. The left hand side square below is a pushout square by definition of the quotient X/A, and it explains how this space becomes pointed:

$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow{\scriptstyle p} & & \big\downarrow \\ \star & \longrightarrow & X/A \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{map}(X/A, Y) & \longrightarrow & \mathrm{map}(\star, Y) \\ \big\downarrow & & \big\downarrow{\scriptstyle p^*} \\ \mathrm{map}(X, Y) & \longrightarrow & \mathrm{map}(A, Y) \end{array}$$

Therefore the righthand square is a pullback by Lemma 4.2. The projection map p induces on mapping spaces the inclusion of constant maps via the identification $\mathrm{map}(\star, Y) \approx Y$. A point $y \in Y$ corresponds to the map sending $\star$ to y, and the latter is sent by p^* to the composition $A \xrightarrow{p} \star \xrightarrow{y} Y$.

The pullback consists then of pairs $(f, y) \in \mathrm{map}(X, Y) \times Y$ such that $f|_A = c_y$. The space of such pairs is homeomorphic to a subspace of $\mathrm{map}(X, Y)$ and when $y = y_0$ is the basepoint, we get precisely the desired subspace. □

A particular case of a quotient space is the smash product $X \wedge Y = X \times Y / X \vee Y$.

Corollary 4.4. *Let X, Y and Z be locally compact Hausdorff spaces. Then $\mathrm{map}_*(X \wedge Y, Z)$ is homeomorphic to the subspace of $\mathrm{map}(X \times Y, Z)$ of all maps sending the wedge $X \vee Y$ to the base point $z_0 \in Z$.* □

We finally arrive to the pointed exponential law.

Theorem 4.5. *Let X, Y and Z be locally compact Hausdorff spaces. Then we have a homeomorphism $\mathrm{map}_*(X \wedge Y, Z) \approx \mathrm{map}_*(X, \mathrm{map}_*(Y, Z))$, natural in Y.*

Proof. By Corollary 4.4 we may identify $\mathrm{map}_*(X \wedge Y, Z)$ with the subspace of $\mathrm{map}(X \times Y, Z)$ of all maps f sending the wedge $X \vee Y$ to the base point $z_0 \in Z$. Under the unpointed exponential law this subspace corresponds to a subspace of $\mathrm{map}(X, \mathrm{map}(Y, Z))$ namely those $a(f)$ such that $a(f)(x) = f(x, -)$ sends y_0 to the base point z_0 for all $x \in X$, i.e., $a(f)(x)$ is a pointed map, and $a(f)(x_0)$ is the constant map c_{z_0}, i.e. the adjoint map $a(f)$ itself is a pointed map. Thus the unpointed adjunction a restricts to a map

$$\mathrm{map}_*(X \wedge Y, Z) \to \mathrm{map}_*(X, \mathrm{map}_*(Y, Z))$$

The inverse unpointed homeomorphism restricted to $\mathrm{map}_*(X, \mathrm{map}_*(Y, Z))$ provides the inverse map, establishing the pointed version. □

Corollary 4.6. *The smash product $X \wedge -$ converts pushouts into pushouts, the pointed mapping space $\mathrm{map}(Y, -)$ converts pullbacks into pullbacks, and the pointed mapping space $\mathrm{map}_*(-, Z)$ converts pushouts into pullbacks.*

Proof. The first statements are direct consequences from the fact that the functors are respectively left and right adjoints. For the third one, beware of the contravariance. Consider a pushout square $\square$ and $\mathrm{map}_*(\square, Z)$. To prove it is a pullback we have to verify the universal property for a diagram of maps out of X. By adjunction this corresponds to a diagram of maps from $X \wedge \square$ into Z. As we have proved above that this square is also a pushout square we can conclude by adjoining back. □

REMARK 4.7. In this section we have followed a traditonal approach to prove the pointed adjunction, relying on the previously established result on unpointed maps. Surprisingly, a direct proof of the adjunction $- \wedge Y \dashv \mathrm{map}_*(Y, -)$ has appeared in print in 1996 only to my knowledge, in an article by Cagliari, [**2**].

5. Loop and suspension

In the final short section of this chapter we specialize to the case when the space Y is the circle.

DEFINITION 5.1. Let X be a pointed space. The *loop space* ΩX is the space of pointed maps $\mathrm{map}_*(S^1, X)$.

DEFINITION 5.2. The *reduced suspension* SX of a pointed space X is the smash product $X \wedge S^1$.

To be very explicit the smash product $X \wedge S^1$ is by definition the quotient space $X \times S^1 / X \vee S^1$ and since S^1 itself is the quotient $I/0 \sim 1$, we identify SX with the quotient

$$X \times I/(x,0) \sim (x,1), (x_0) \sim (x_0, 0), (x_0, t) \sim (x_0, 0)$$

Taking this double quotient in the reverse order we see that SX is homeomorphic to the quotient $\Sigma X / x_0 \times I$. For a well-pointed space this quotient obtainied by contracting a segment does not change the homotopy type and hence $\Sigma X \simeq SX$. The advantage of the latter is that it has a well-defined and canonical base point, namely the class of $x_0 \times I$.

PROPOSITION 5.3. *There is a natural homeomorphism* $\mathrm{map}_*(X, \Omega Y) \approx \mathrm{map}_*(SX, Y)$.

COROLLARY 5.4. *There is a bijection* $[X, \Omega Y]_* \cong [SX, Y]_*$.

PROOF. Since paths in a mapping space correspond to homotopies we apply π_0, the set of path connected components to the homeomorphism in Proposition 5.3. $\square$

CHAPTER 2

Homotopy groups

We introduce higher homotopy groups for pointed spaces, as a generralization of the fundamental group, a.k.a. the first homotopy group. We show that these higher analogs are always abelian, in two different ways. One classical way describes explicit homotopies between the sums $f+g$ and $g+f$, the other is more categorical in nature, we introduce the notion of (co)-H-spaces and (c)-H-groups. We also establish the existence of long exact sequences of homotopy groups, similar to the long exact sequences in homology we already know.

1. Higher homotopy groups

We have seen in the Topology course that $\pi_0 X$, the set of path connected components of a pointed space (X, x_0) can be identified with the set of pointed homotopy classes of maps $[(S^0, 1); (X, x_0)]_*$, which we often write $[S^0, X]_*$ for short when the base point is understood from the context. We also identified the *group* of homotopy classes of based loops, where the group law is induced by concatenation of loops, with $[S^1, X]_*$. We have also reinterpreted concatenation in a more diagrammatic way using the pinch map on the circle $p\colon S^1 \to S^1 \vee S^1$. This is nothing but the quotient map that identifies 1 and -1, and if we are given two loops $f, g\colon S^1 \to X$, the product of their homotopy classes is represented by

$$S^1 \xrightarrow{p} S^1 \vee S^1 \xrightarrow{f \vee g} X \vee X \xrightarrow{\nabla} X$$

where ∇ is the fold map.

Remark 1.1. Any group G can be realized as fundamental group of a space. This is a consequence of the Seifert-van Kampen Theorem since we can choose a presentation of G by generators g_α and relators r_β living in the free group $F(g_\alpha)$. The fundamental group of the wedge $\bigvee_\alpha S^1$ is isomorphic to this free group and we just need to attach one 2-cell for each relator $r_/beta$ so as to construct a space

$X = (\bigvee_\alpha S^1) \cup (\bigcup_\beta e^2)$ whose fundamental is isomorphic to the quotient of $F(g_\alpha)$ by the normal subgroup generated by the r_β's, i.e. G.

Definition 1.2. Let (X, x_0) be a pointed space and $n \geq 1$. The *n-th homotopy group* $\pi_n X = \pi_n(X, x_0) = [S^n, X]_*$.

It is also the pinch map, on S^n which justifies the name and yields a group structure. We also denote by p the collapse map by the equatorial sphere S^{n-1} in S^n.

Lemma 1.3. *Let X be a pointed space. The pinch map on S^n induces a group structure on $\pi_n X$. For two pointed maps $f, g\colon S^n \to X$ the sum $[f]+[g]$ is represented by the the composition $f \star g\colon S^n \xrightarrow{p} S^n \vee S^n \xrightarrow{f \vee g} X \vee X \xrightarrow{\nabla} X$.*

Proof. This goes exactly the same way as when $n = 1$. The constant map c_{x_0} represents the neutral element because the collapse of an hemisphere $D^n_+ \subset S^n$ is homotopic to the identity up to identifying the quotient S^n/D^n_+ with S^n. The operation $\star$ is associative up to homotopy, hence strictly associative on homotopy classes of maps. This follows from the observation (analogous to what we have done for the fundamental group by looking at trisections of an interval) that pinching the tropic of cancer and the equator or pinching the tropic of capricorn and the equator are different maps, but they are homotopic, the homotopy moving slowly, in one second, and continuously the latter up by 25.5 degrees.

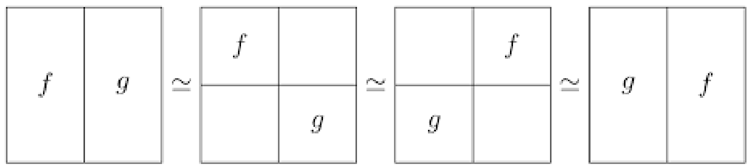
Finally, given a pointed map $f\colon S^n \to X$, let us view S^n as the reduced suspension SS^{n-1}, whose elements are classes $[x, t]$ for $x \in S^{n-1}$ and $t \in [0, 1]$. The inverse of $[f]$ is then represented by the map $[x, t] \mapsto f[x, 1 - t]$. □

Here comes the first proof of the commutativity of higher homotopy groups. Instead of providing explicit formulas for the homotopies we show hopefully helpful drawings in the case $n = 2$.

Proposition 1.4. *For any $n \geq 2$ $\pi_n X$ is abelian and we write $+$ for $\star$.*

Proof. We represent a map from a sphere S^2 by a map from a square that is constant on its boundary (it factors thus through the quotient $(I \times I)/\partial(I \times I) \approx S^2$). Therefore the map $f \star g$ is picturally described by the first drawing below, even though

I would prefer to see f on top of g since my model for the suspension has a vertical interval in my mind, but stealing the drawing from Najib Idrissi on stackexchange does not allow me to complain:



The parts of the drawing with an f or a g inside indicate simply a reparametrization of the map on a homeomorphic square or rectangle, the parts without a letter support a constant map. □

We do not provide any proofs yet, but indicate a few elementary computations.

EXAMPLE 1.5. (1) $\pi_2 S^2 \cong \mathbb{Z}$, where a generator is the homotopy class of the identity on the sphere;

(2) $\pi_n S^1 = 0$ for all $n \geq 2$ using the theory of covering spaces;

(3) $\pi_3 S^2 \cong \mathbb{Z}$, where a generator is the Hopf map $\eta\colon S^3 \to S^2$.

2. H-spaces and co-H-spaces

We follow [**8**, Section 7.2]. To prove the previous proposition, there is a categorical argument, called the Eckmann-Hilton argument (see also Topologie, Série 7, for a version of this trick).

DEFINITION 2.1. A pointed space (X, x_0) is an *H-space* if it is equipped with a *multiplication* map $m\colon X \times X \to X$ such that the following diagram commutes up to pointed homotopy:

$$\begin{array}{ccccc} X & \xrightarrow{i_1} & X \times X & \xleftarrow{i_2} & X \\ & {\scriptstyle id}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle id} & \\ & & X & & \end{array}$$

This means that we dot not require the multiplication to be strict, but the base point plays the role of a neutral element, up to homotopy. For the moment we do not require m to be associative, even up to homotopy. But these are features an H-space could have of course.

Definition 2.2. An H-space X is *homotopy associative* if $m \circ (m \times X) \simeq m \circ (X \times m)$ and it is *homotopy commutative* if $m \circ T \simeq m$ where $T\colon X \times X \to X \times X$ is the interchange map $T(x, y) = (y, x)$.

In terms of diagrams homotopy associativity means that

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{X \times m} & X \times X \\ {\scriptstyle m \times X}\downarrow & & \downarrow{\scriptstyle m} \\ X \times X & \xrightarrow[m]{} & X \end{array}$$

commutes up to pointed homotopy and homotopy commutativity means that the following triangle commutes up to pointed homotopy:

$$\begin{array}{ccc} X \times X & \xrightarrow{T} & X \times X \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & X & \end{array}$$

Definition 2.3. A homotopy associative H-space X is an *H-group* if there is an *inverse map* $\iota\colon X \to X$ such that the composition $X \xrightarrow{\Delta} X \times X \xrightarrow{X \times \iota} X \times X \xrightarrow{m} X$ is homotopically constant.

Hence, in an H-group $m(x, \iota x)$ is not equal to the base point x_0, but continuously deformable into c_{x_0}.

Among the following examples, the first two are historically significant, the last one is important in this course.

Example 2.4.
(1) Any topological group is an H-group. For example $S^1 = U(1) \cong SO(2)$, $S^3 = SU(2)$, or other compact Lie groups like $SO(n), SU(n)$, are H-groups.
(2) The only other sphere, except S^0, S^1, S^3, which can be given the structure of an H-space is S^7, the unit octonionic sphere, but it fails to be an H-group because the multiplication is no associative. We will not be able to give a proof of this deep theorem in this course.
(3) Any loop space $\Omega X = \operatorname{map}_*(S^1, X)$ is an H-group. Multiplication is concatenation of loops, which is homotopy associative as we have seen in the

Topology class, the inverse map ι corresponds to reversing the parametrization of loops. Notce that ΩX is not homotopy commutative in general because of the next observation.

LEMMA 2.5. *Let X be an H-group. Then $\pi_0 X$ inherits a group structure, which is commutative if X is homotopy commutative.*

PROOF. The group axiom hold up to pointed homotopy in X, they thus hold strictly in $\pi_0 X$. □

This lemma is just a particular case of the following result (we have not used at all that the source is S^0 above).

PROPOSITION 2.6. *Let X be an H-group. Then $[W, X]_*$ inherits a group structure $\circ$ for any pointed space W, which is commutative if X is homotopy commutative.*

PROOF. The multiplication m induces product $\circ$ on $[W, X]_*$. Given two pointed maps $f, g \colon W \to X$, we define the product $f \circ g$ by

$$W \xrightarrow{\Delta} W \times W \xrightarrow{f \times g} X \times X \xrightarrow{m} X$$

We also write $[f] \circ [g]$ for the product defined on homotopy classes. □

REMARK 2.7. *There is a notion of H-map betwen H-spaces, where we require the obvious compatibility between the multiplications.*

The whole theory dualizes from multiplications to comultiplications.

DEFINITION 2.8. A pointed space (X, x_0) is an *co-H-space* if it is equipped with a *comultiplication* map $\psi \colon X \to X \vee X$ such that the following diagram commutes up to pointed homotopy:

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{id} & \downarrow \psi & \searrow^{id} & \\ X & \xleftarrow[p_1]{} & X \vee X & \xrightarrow[p_2]{} & X \end{array}$$

The maps p_1 and p_2 collapse respectively the second and first wedge summand to the base point. In terms of universal properties, p_1 is the unique map whose restriction to the first wedge summand is the identity and the restriction to the second is constant. In terms of explicit formulas, seeing $X \vee X$ as a subspace of $X \times X$ the map p_2 is defined by $p_2(x; x_0) = x_0$ and $p_2(x_0; x) = x$, so its the restriction of the projection onto the second factor.

The analogous notions of homotopy coassociative, homotopy cocommutative, co-H-group, are easy to formulate and are left to the reader.

Proposition 2.9. *Let X be a co-H-group. Then $[X, Z]_*$ inherits a group structure $\star$ for any pointed space Z, which is commutative if X is homotopy cocommutative.*

Proof. The comultiplication ψ induces product $\star$ on $[X, Z]_*$. Given two pointed maps $f, g \colon X \to Z$, we define the product $f \star g$ by

$$X \xrightarrow{\psi} X \vee X \xrightarrow{f \vee g} Z \vee Z \xrightarrow{\nabla} Z$$

We also write $[f] \star [g]$ for the product defined on homotopy classes. □

Our main source of examples for co-H-spaces is given by the reduced suspension.

Example 2.10. Let $SX = S^1 \wedge X$ be the reduced suspension of any pointed space X. The pinch map defines a comultiplication

$$S^1 \wedge X \xrightarrow{p \wedge X} (S^1 \vee S^1) \wedge X \approx (S^1 \wedge X) \vee (S^1 \wedge X)$$

We have used here that $- \wedge X$ is a left adjoint on pointed spaces, thus it preserves pushouts, in particular wedges.

The inverse ι is induced by reversing the parametrization $S^1 \to S^1$, which sends e^{it} to e^{-it}.

3. The Eckmann-Hilton argument

When X is a co-H-space and Y is an H-space, one can define two a priori different group structures on $[X, Y]_*$, which we denoted by $\circ$ and $\star$ in the previous section. We will prove that they coincide and give the set of pointed homotopy classes the structure of an abelian group.

Here comes the famous Eckmann-Hilton Lemma.

LEMMA 3.1. *Let $(G, 1)$ be a set equipped with two unital composition laws $\circ$ and $\star$ that satisfy the interchange law*

$$(a \star b) \circ (c \star d) = (a \circ c) \star (b \circ d)$$

for all $a, b, c, d \in G$. Then $\circ = \star$ and $a \circ d = d \circ a$ for all $a, d \in G$.

PROOF. We perform two simple computations where we will see the importance of the fact that both laws share the same neutral element. First

$$a \circ d = (a \star 1) \circ (1 \star d) = (a \circ 1) \star (1 \circ d) = a \star d$$

shows that $\circ$ and $\star$ coincide. Second

$$a \circ d = (1 \star a) \circ (d \star 1) = (1 \circ d) \star (a \circ 1) = d \star a = d \circ a$$

which concludes the proof. □

THEOREM 3.2. *Let X be a co-H-group and Y be an H-group. The group structures they define on $[X, Y]_*$ coincide and are abelian.*

PROOF. We have already seen that the homotopy class of the constant map c_{y_0} is a common unit, so we only to check that $\star$ and $\circ$ satisfy the interchange law and conclude by the Eckmann-Hilton argument, see Lemma 3.1. Consider thus four pointed maps $a, b, c, d \colon X \to Y$ and the following diagram allowing us to compare both fourfold products:

$$\begin{array}{ccccccccc}
X & \xrightarrow{\psi} & X \vee X & \xrightarrow{\Delta \vee \Delta} & (X \times X) \vee (X \times X) & \xrightarrow{a \times c \vee b \times d} & (Y \times Y) \vee (Y \times Y) & \xrightarrow{m \vee m} & Y \vee Y \\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} & & & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \nabla} \\
X \times X & \xrightarrow[\psi \times \psi]{} & (X \vee X) \times (X \vee X) & \xrightarrow[a \vee b \times c \vee d]{} & (Y \vee Y) \times (Y \vee Y) & \xrightarrow[\nabla \times \nabla]{} & Y \times Y & \xrightarrow[m]{} & Y
\end{array}$$

By definition of $\star$ and $\circ$ the upper composite of solid arrows going all the way from X to the bottom right Y represents $(a \star c) \circ (b \star d)$ whereas the bottom composition represents $(a \circ b) \star (c \circ d)$.

Let us complete the diagram by adding the two vertical dashed arrows. They render the left, respectively right, square strictly commutative (which has nothing to do with the fact that ψ and m are (co)multiplications). We are thus left with the middle rectangle.

Both ways to compose maps correspond to maps from a wedge $X \vee X$ to a product $Y \times Y$. By the universal property for a coproduct in the category of pointed spaces, it is thus enough to compare the restriction to both wedge summands. Let us follow for example a point of the form (x, x_0):

$$\nabla \circ (a \times c \vee b \times d) \circ (\Delta \vee \Delta)(x, x_0) = \nabla \circ (a \times c \vee b \times d)(x, x, x_0, x_0) = \nabla(a(x), c(x), b(x_0), d(x_0))$$

Since b and d are pointed maps, this is $\nabla(a(x), c(x), y_0, y_0) = (a(x), c(x))$. Let us finally follow the other comnposition through the left and bottom part of the rectangle:

$$(\nabla \times \nabla) \circ (a \vee b \times c \vee d) \circ \Delta((x, x_0) = (\nabla \times \nabla) \circ (a \vee b \times c \vee d)((x, x_0, x, x_0) = (\nabla \times \nabla)(a(x), y_0, c(x), y_0)$$

Again we find $(a(x), c(x))$, on the nose! We have thus shown that the diagram commutes strictly and this finishes the proof. □

Let us pause here to remark that the strict equality of the interchange law, before passing to homotopy classes of maps, does not imply that we have a group structure on the mapping space $\mathrm{map}_*(X, Y)$ or even the set of morphisms $\mathrm{mor}_*(X, Y)$ since the unit and associativity do not hold strictly, but only up to homotopy.

The following corollaries are now consequences of this very general principle.

Corollary 3.3. *Let X and Y be two pointed spaces. The two group structures on $[SX, \Omega Y]_*$ coming from the pinch map on SX and loop concatenation on ΩY coincide and are abelian.*

Corollary 3.4. *Let X be a pointed spaces. For any $n \geq 2$ the group $\pi_n X$ is abelian.*

Proof. We use the previous result for $S^{n-1} = SS^{n-2}$ and the loop-suspension adjunction. □

4. Relative homotopy groups

Just like homology groups, homotopy groups admit a relative version for pairs of spaces (X, A) where $A \subset X$ is a subspace of X containing the base point $x_0 = a_0$. We will establish the existence of a long exact sequence and start with a warning about the model we use. Homotopy classes of maps from spheres to some space are replaced by homotopy classes of maps from the pair (D^n, S^{n-1}) but some authors prefer the homeomorphic pair $(I^n, \partial I^n)$ which has some advantages related to the parametrization.

Definition 4.1. Let (X, A) be a pair of pointed spaces and $n \geq 1$. The *n-th relative homotopy group* $\pi_n(X, A)$ is equal to the set of pointed homotopy classes of pairs $[(D^n, S^{n-1}), (X, A)]_*$.

Thus, elements in $\pi_n(X, A)$ are represented by pointed maps $f\colon D^n \to X$ whose restriction to S^{n-1} land into A. Homotopies are to be taken in the pointed sense and restrict to homotopies from $S^{n-1} \times I$ into A.

The pinch map on S^{n-1} extends to a map on the n-dimensional ball D^n by collapsing the whole equatorial disc. We need $n \geq 2$ to do so.

Lemma 4.2. *The pinch map on D^n gives $\pi_n(X, A)$ a group structure for $n \geq 2$.*

Relative homotopy groups generalize the notion of (absolute) homotopy groups.

Lemma 4.3. *For any $n \geq 1$ we have a bijection $\pi_n(X, x_0) \cong \pi_n X$.*

Proof. A map of pairs $f\colon (D^n, S^{n-1}) \to (X, x_0)$ is a map which is constant on S^{n-1} so that it corresponds, by the universal property of the quotient to a pointed map $\bar{f}\colon D^n/S^{n-1} \approx S^n \to X$. Likewise relative homotopies which are constant on S^{n-1} factor through $(D^n/S^{n-1}) \times I$. □

We have already met the category of pairs in an exercise last week, so the next proposition should not come as a surprise.

Proposition 4.4. *For any $n \geq 1$ the n-th relative homotopy group is a functor from the category of pairs to the category of pointed sets, or groups if $n \geq 2$.*

Proof. A realtive pointed map $f\colon (X, A) \to (Y, B)$ induces a map on relative homotopy groups by (post)composition. □

In order to set up the long exact sequence for homotopy groups, we introduce a homomorphism connecting homotopy groups in two adjacent degrees.

DEFINITION 4.5. Let (X, A) be a pair of pointed spaces and $n \geq 1$. The *connecting homomorphism* $\partial\colon \pi_n(X, A) \to \pi_n A$ sends the homotopy class of a map of pairs $a\colon (D^n, S^{n-1}) \to (X, A)$ to the homotopy class of the restriction $a\mid_{S^{n-1}}$.

REMARK 4.6. This conencting homomorphism is well-defined since by definition a homotopy H of pairs $a \simeq a'$ restricts to a homotopy on $S^{n-1} \times I$ whose image lies in the subspace A.

When $n \geq 2$ this map is indeed a homomorphism because the pinch map on the ball D^n restricts to the pinch map on its boundary sphere. However, when $n = 1$ we are looking at $\pi_1(X, A)$, (pointed) homotopy classes of maps $a\colon (I, \{0; 1\}) \to (X, A)$, i.e. paths in X starting at the base point and whose end point belongs to A. The connecting morphism sends a to its end point $a(1)$, or rather its class in $\pi_0 A$.

Before proving that the connecting homomorphisms are part of a long exact sequence, we establish the so-called *Compression Lemma*, which is interesting in its own right since it tells us what it means for a map of pairs to be homotopically trivial, which is not as transparent as in the absolute setting. We say that two maps $f, g\colon X \to Y$ are homotopic *relative to a subspace* $A \subset X$ is there exists a homotopy $H\colon X \times I \to Y$ from f to g which is constantly equal to $f\mid_A = g\mid_A$ during the homotopy, i.e. $H(a, t) = f(a)$ for all $a \in A$. This is stronger than to require that we have a homotopy of pairs for maps out of (X, A).

LEMMA 4.7. *A map* $a\colon (D^n, S^{n-1}) \to (X, A)$ *represents the neutral element in* $\pi_n(X, A)$ *if and only if it is homotopic to a map, relative to* S^{n-1}*, whose image lies entirely in* A.

PROOF. If a is homotopic to a map $b\colon D^n \to A \subset X$, then, by contractibility of D^n, we can retract D^n to its base point so that $[b] = [c_{x_0}]$.

Conversely, assume that a is homotopic to the constant map c_{x_0} in the relative sense, i.e., there exists a homotopy $H\colon D^n \times I \to X$ such that $H(-, 0) = a$, $H(s, 1) = x_0$ for all $s \in D^n$ and H restricts on $S^{n-1} \times I$ to a homotopy entirely contained in A.

Consider the subspaces $D_t = D^n \times t \cup S^{n-1} \times [0, t] \subset D^n \times [0, 1]$. For all $0 \leq t \leq 1$ they are homeorphic to a disc whose boundary is $S^{n-1} \times 0$. Viewing the homotopy H as a continuous deformation from $a = H\mid_{D_0}$ to $b = H\mid_{D_1}$ we have indeed a homotopy from a to a map b which is constantly equal to $a\,|_{S^{n-1}}$ on the boundary. The image of b is entirely contained in A. □

To fix some notation let us write $i\colon A \to X$ for the subspace inclusion of a pair of pointed spaces (X, A), and $j\colon (X, x_0) \to (X, A)$ for the inclusion of pairs. When talking about exact sequences of pointed sets we simply mean that "the image equals the kernel", or more precisely that the image consists of homotopy classes of maps that are sent to the homotopy class of the constant map.

THEOREM 4.8. *For any pair of pointed spaces (X, A) there is a long exact sequence*

$$\cdots \to \pi_{n+1}(X, A) \xrightarrow{\partial} \pi_n A \xrightarrow{i_*} \pi_n X \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\partial} \cdots \to \pi_0 X$$

where the last three homotopy classes only form pointed sets.

PROOF. **Part 1.** We prove first exactness at $\pi_n X$. Since $j_* \circ i_* = (j \circ i)_*$ we deduce from the Compression Lemma 4.7 that this composition is zero: Indeed, given a map $a : (D^n, S^{n-1}) \to (A, x_0)$, the composition with $(A, x_0) \subset (X, A)$ yields precisely a map whose image lies in A.

To show that $\mathrm{Ker} j_* = \mathrm{Im} i_*$ consider now a map $b\colon (D^n, S^{n-1}) \to (X, x_0)$ such that $j_*[b] = 0$. By the Compression Lemma again this means that $j \circ b$ is homotopic, relative to S^{n-1} to a map $b'\colon (D^n, S^{n-1}) \to (X, A)$ whose image lies in A, and since the homotopy is constant on S^{n-1}, also $b'\,|_{S^{n-1}}$ is constant. In particular b' can be seen as a map $a\colon (D^n, S^{n-1}) \to (A, x_0)$. Then $i \circ a = b'$ so $i_*[a] = [b'] = [b]$.

Part 2. We move to exactness at $\pi_n(X, A)$ and compute $\partial \circ j_*$. Given a map of pairs $b\colon (D^n, S^{n-1}) \to (X, x_0)$, by Definition 4.5 of the connecting homomorphism, the class $\partial(j_*[b])$ is represented by the restriction $j \circ b\,|_{S^{n-1}}$, which is constant. So $\partial(j_*[b]) = 0$.

To prove exactness we consider now a map $f\colon (D^n, S^{n-1}) \to (X, A)$ and assume that $\partial[f] = 0$, i.e. $f\,|_{S^{n-1}}$ is nullhomotopic as pointed map to A, via a homotopy $F\colon S^{n-1} \times I \to A$. We define a new map g on $D^n \approx D^n \times 1 \cup S^{n-1} \times I$ by using F on the cylinder and our map f on $D^n \times 1$. Observe that f and g are homotopic as maps of pairs since F takes place entirely inside A. But g is map which is constant on its boundary, it defines thus a map $(D^n, S^{n-1}) \to (X, x_0)$. Now $j_*[g] = [f]$.

Part 3. We end with the exactness at $\pi_{n-1}A$. Given $f\colon (D^n, S^{n-1}) \to (X, A)$, the class $i_*(\partial[f])$ is represented by the restriction $f\mid_{S^{n-1}}$ composed with i. But f itself can be seen as a nullhomotopy for this map. Therefore $i_* \circ \partial = 0$.

To conclude let us consider a map $a\colon S^{n-1} \to A$ with $i_*[a] = 0$, i.e. $i \circ a$ is homotopic to the constant map c_{x_0} via a homotopy H. As usual such a nullhomotopy can be parametrized by a map on D^n, by collapsing $S^{n-1} \times 1$ to a point. This gives us a map $\overline{H}\colon D^n \to X$ whose restriction to S^{n-1} is equal to the map a we started with. Considering $\overline{H}$ as a map of pairs we have $\partial[H] = [a]$. □

Example 4.9. Let CX be the cone on a space X and consider the long exact sequence in homotopy for the pair (CX, X):

$$0 = \pi_2 CX \to \pi_2(CX, X) \xrightarrow{\partial} \pi_1 X \to \pi_1 CX \to \pi_1(CX, X) \xrightarrow{\partial} \pi_0 X \to \pi_0 CX = *$$

We conclude from the contractibility of CX that the connecting homomorphism induces an isomorphism $\pi_{n+1}(CX, X) \cong \pi_n X$ for all $n \geq 1$. At the end of the sequence we have to be more careful because this is only an exact sequence of sets. The map of sets $\partial\colon \pi_1(CX, X) \to \pi_0 X$ is surjective (its image coincides with the kernel of the "zero map". We also know that ∂ has its kernel reduced to the class of the trivial map $(D^1, S^0) \to (CX, X)$. Let us see if it is a bijection. The elements of $\pi_1(CX, X)$ are homotopy classes of paths $I \approx D^1 \to CX$ whose endpoint lies in X. Choosing a point x in a connected component of X we claim that any two paths in CX ending at x are homotopic as relative maps. For the connected component of the base point this follows from the "injectivity", and for other components any such relative path must go through the top of the cone.

Example 4.10. Let X be a reduced CW-complex (having a single 0-cell), so its 1-skeleton $X^{(1)}$ is a wedge of circles $\vee_\alpha S^1$. A wedge of circles has trivial higher homotopy groups (because its universal cover is contractible, it is a Cayley graph of a free group). The long exact sequence in homotopy then shows that $\pi_n X \cong \pi_n(X, X^{(1)})$ for any $n \geq 3$ and let us again look more closely at the end of the sequence:

$$0 \to \pi_2 X \cong \pi_2(X, X^{(1)}) \xrightarrow{\partial} \pi_1 X^{(1)} \to \pi_1 X \to \pi_1(X, X^{(1)}) \to *$$

Here $\pi_1 X^{(1)}$ is a free group projecting onto $\pi_1 X$ so $\pi_1(X, X^{(1)}) = *$.

CHAPTER 3

The cellular approximation

In this chapter we will show that any map between CW-complexes can be deformed, up to homotopy, into a map which is cellular: it sends n-dimensional cells to the n-skeleton. There will be a technical part in the proof, but as a reward we will be able to deduce the value of some very important homotopy groups of spheres, and other highly connected complexes. We follow quite closely [**3**, Chapter 4], which has almost been covered in Christian Urech's course "Algebraic Topology", so we should be quite familiar with the notation and the techniques.

1. CW-complexes and polyhedra

We write $X^{(n)}$ for the n-skeleton of a CW-complex X.

DEFINITION 1.1. Let X, Y be CW-complexes. A map $f\colon X \to Y$ is *cellular* if $f(X^{(n)}) \subset Y^{(n)}$ for any $n \geq 0$.

The following example already shows why it will be quite handy to be able to deform any map into a cellular one.

EXAMPLE 1.2. Let us use the *standard* cell structure on $S^n = e^0 \cup e^n$, where the attaching map for the n-cell is the only one there is, namely the constant map $S^{n-1} \to e^0 = *$. This means that this model for an n-dimensional sphere has all its skeleta $(S^n)^{(k)}$ reduced to a point for $k < n$.

Let $m > n$. A map $f\colon S^n \to S^m$ is cellular if and only if it is constant.

The main technique to control what a map does on a cell will be to homotope it to a piecewise linear map on a polyhedron. For us polyhedra will always be finite, so we do not mention it explicitly in the terminology.

DEFINITION 1.3. A *convex polyhedron* in $\mathbb{R}^n$ is a finite intersection of half-spaces whose boundaries are hyperplanes defined by an affine equation $\sum a_i x_i = b$.

A subspace $K \subset \mathbb{R}^n$ is a *polyhedron* if it is a finite union of compact convex polyhedra.

EXAMPLE 1.4. A quadrant in $\mathbb{R}^2$ is a convex polyhedron, but it is not compact, so we will not work with such polyhedra here. Any triangle in $\mathbb{R}^2$ is compact convex polyhedron, a not necessarily convex hexagon is a polyhedron. The standard simplex in $\mathbb{R}^{n+1}$, convex hull of the endpoints of the standard basis, is a polyhedron.

DEFINITION 1.5. Let K be a polyhedron. A map $f\colon K \to \mathbb{R}^k j$ is *piecewise linear*, or PL for short, if there exists a decomposition of K into compact convex polyhedra $K_i \subset K$ such that $f\,|_{K_i}$ is affine.

A map $f\colon K \to \mathring{e}^k$ is *piecewise linear* if it so after composing with some homeomorphism $\mathring{e}^k \approx \mathbb{R}^k$.

EXAMPLE 1.6. Every cube I^n is a polyhedron and it can be given the structure of a simplicial complex, either by taking barycentric subdivision, or by doing it inductively, starting from the segment I which is a simplicial complex, continuing with the square, which can be cut into two triangles diagonally, etc.

For further use we introduce now a construction.

CONSTRUCTION 1.7. Let $f\colon I^n \to \mathbb{R}^k$ be any map. Consider two closed balls $B_1 = \overline{B(0;1)} \subset B_2 = \overline{B(0;2)}$ in $\mathbb{R}^k$. Since f is continuous on a compact cube, it is uniformly continuous. There exists $\varepsilon > 0$ such that

(1) If $|x-y| < \varepsilon$ then $|f(x) - f(y)| < 1/2$;
(2) $\varepsilon < \frac{1}{2} d(f^{-1}(B_1); I^n \setminus f^{-1}(\mathring{B}_2))$

Choose an integer N such that the diameter of the small cube $[0;1/N]^n$ is smaller than ε and subdivide the big cube I^n into small cubes of side $1/N$. Set K to be the union of all small cubes meeting $f^{-1}(B_1)$ and make it somewhat fatter by setting K_2 to be the union of all cubes meeting K.

Observe that every point of K_2 is at a distance from K smaller than the diameter of a small cube, so smaller than ε. Then, since K contains the preimage of B_1, the distance to $f^{-1}(B_1)$ is smaller than 2ε. Moreover, by choice of ε, the polyhedron K_2 is containes in $f^{-1}(B_2)$.

2. Cellular approximation

Our proof will go by induction on the cells in the source, so we need first a result about a map from one cell. All technical issues are contained in this result. We will need to deal with maps which do not miss any point in the interior of large cells (if there is a point not in the image of f one can retract the cell down to its boundary).

LEMMA 2.1. *Let $f\colon I^n \to Z = W \cup e^k$ be a map where the interior $\mathring{e}^k$ is entirely contained in the image of f. Then f is homotopic to a map f_1, relative to $f^{-1}(W)$, such that there exists a polyhedron $K \subset I^n$ with*

(a) $f_1(K) \subset e^k$ and $f\,|_K$ is PL;
(b) There is a non-empty open subset $U \subset e^k$ such that $f_1^{-1}(U) \subset K$.

PROOF. Let us fix a homeomorphism $\mathring{e}^k$ and apply Construction 1.7 to the unit ball B_1, the larger ball B_2 in $\mathbb{R}^k$. This allows us to find a polyhedron $K \subset K_2$ in I^n on which we will define a PL map g and then homotope it to f using the extra space between K and K_2.

Since K_2 is made of little cubes, we can use Example 1.6 to view it as simplicial complex, i.e. a union of standard simplices. We define the map g by starting on the vertices (0-simplices) by using the map f, so we set $g(v) = f(v)$ for any $v \in (K_2)^{(0)}$. We then extend the map linearly to all simplices of K_2. This map g coincides with f on vertices, but not on the whole boundary of K_2 so we need to alter it between K and this boundary to be able to glue it with f on the complement of K_2.

To do so we find first a map $\varphi\colon K_2 \to [0;1]$ which is constant, equal to 1 on K and zero on ∂K_2. We can use the same technique as above to make it continuous by extending it linearly. Next we construct a homotopy

$$H\colon K_2 \times I \to e^k, (x;t) \mapsto (1 - t\varphi(x)) \cdot f(x) + t\varphi(x) \cdot g(x))$$

At time $t = 0$ we start with $H(x, 0) = f(x)$ and at the other end, $t = 1$ we have $H(x, 1) = (1 - \varphi(x)) \cdot f(x) + \varphi(x) \cdot g(x))$, a map which coincides with g on K. As mentioned above the last important property is that for $x \in \partial K_2 \setminus K$ we have $H(x, t) = f(x)$ since $\varphi(x) = 0$ here. This means that we can extend H continuously to a homotopy on the whole cube I^n which is constantly equal to f outside K_2. This yields a homotopy H' from f to a map $f_1 = H'(-, 1)$ which is PL on K.

Let us now verify the properties (a) and (b). Property (a) is immediate by construction, so we need only to find the open set U from claim (b). Consider the compact subspace $C = f_1(\overline{I^n \setminus K}) \subset W \cup e^k$.

Claim. The point 0, i.e., the center of the open disc $\mathring{e}^k$ we have identified with $\mathbb{R}^k$, does not belong to C.

Proof of the Claim. On $I^n \setminus K_2$ the map f_1 coincides with f and the latter sends a point there outside B_1 since $f^{-1}(B_1) \subset K \subset K_2$. For points on the boundary, we proceed as follows. Any maximal dimensional n-simplex σ in K_2 but not in K is contained in a small cube of side $1/N$, hence its image $f(\sigma)$ is contained in a ball of radius $1/2$ by choice of ε. However, to construct g we kept the same value as f on vertices and since a ball is convex, $g(\sigma)$ is contained in the same ball of radius $1/2$. In fact the formula defining H shows that $H(\sigma, t)$ is also contained in that same ball (convex hull). In particular so is $f_1(\sigma)$. The center of this ball is at distance $> 1/2$ from the origin, if not $f(\sigma)$ would be entirely contained in B_1, but we chose σ with vertices outside K. This shows that $0 \notin C$, proving the claim.

We thus choose U to be a neighborhood of 0 such that $f_1(\overline{I^n \setminus K}) \cap U = \emptyset$. This is what we wanted: $f_1^{-1}(U) \subset K$. □

We are ready now for the proof of the *Cellular Approximation Theorem.* We will only do the non-relative version of the following, to keep it as simple as possible, but it does not involve much more to actually prove a version where one deforms a given map f only outside a subcomplex on which the map is already cellular. The main idea is to apply the previous lemma so as to deform a map in such a way that images of small cells do not entirely cover the large cells they meet in the image, because this will allow us to contract those large cells down to their boundary. This should be reminiscent of the proof we saw in Topology that $\pi_1 S^2 = 1$: some loops in S^2 could fill S^2, but up to homotopy we can make it affine in a small neighborhood and since a finite number of affine functions can never fill a square, we chose a point not in the image, prick the sphere at the point and contract it down to a point continuously.

THEOREM 2.2. *Let $f\colon X \to Y$ be a map between CW-complexes. Then f is homotopic to a map g which is cellular. Moreover, if f is cellular on a sub-CW-complex $A \subset X$, then we can choose the homotopy relative to A.*

PROOF. We study this question cell by cell, and to do so, we start with 0-dimensional cells where the solution is easy (we do not need Lemma 2.1).

Each 0-cell is a vertex $v \in X$, sent to a point $f(v)$ that belongs to some path component of Y. A path in this component defines a homotopy from $f\mid_{X^{(0)}}$ to a cellular map if the end point of this path is a vertex in Y. We extend this homotopy, that is only defined on the 0-skeleton, to higher cells by induction, using the fact that the inclusion $S^{n-1} \times I \cup D^n \times 0 \subset D^n \times I$ is a strong deformation retract, so it admits a homotopy inverse $r\colon D^n \times I \to S^{n-1} \times I \cup D^n \times 0$. Let us spell out the details for 1-cells and leave it for later to do this properly for higher cells, since this is a property that so-called cofibrations have in general (we will see that inclusions of subcomplexes are cofibrations). As promised let us look at a 1-cell e^1 attached to the 0-skeleton by an attaching map $a\colon S^0 \to X^{(0)}$. We have a homotopy $H^{(0)}$ defined on $X^{(0)} \times I$ and f itself is already given on $X^{(0)} \cup_a e^1$. Observe that $(X^{(0)} \cup_a e^1) \times I$ is the pushout of

$$(X^{(0)} \cup_a e^1) \times 0 \cup (X^{(0)} \times I) \leftarrow (D^1 \times 0) \cup (S^0 \times I) \hookrightarrow D^1 \times I$$

The universal property of the pushout tells us it is enough to define a map on $D^1 \times I$ compatible with $H^{(0)}$. To do so we use the retraction and compose with the map we already have. We continue by induction on all cells.

Let us thus assume that f is already cellular on $X^{(n-1)}$ and consider the attaching map for a single n-cell e^n, call it again $a\colon S^{n-1} \to X^{(n-1)}$. We will also call f the restriction to $X^{(n-1)} \cup_a e^n$ and have the following situation represented in the following diagram:

$$\begin{array}{ccccc}
S^{n-1} & \xrightarrow{a} & X^{(n-1)} & \xrightarrow{f|_{X^{(n-1)}}} & Y^{(n-1)} \\
\downarrow & & \downarrow & & \downarrow \\
D^n & \longrightarrow & X^{(n-1)} \cup_a e^n & \xrightarrow{f} & Y
\end{array}$$

where the left hand square is a pushout square. Since D^n is compact, so is its image, which thus meets only a finite number of cells in Y. If all these cells have dimension

$\leq n$ there is nothing to do, f would already be cellular, but if not let us consider a maximal dimensional cell e^k in Y that is hit by f, with $k > n$. So the image of f is contained in a subcomplex Z of Y containing $Y^{(n-1)}$, plus a finite number of cells, including e^k. Let us call $W = Z \setminus \mathring{e}^k$ so $Z = W \cup e^k$. If the interior of this cell e^k is not entirely contained in the image of f we can move directly to the second part of the algorithm, but let us suppose first that e^k is entirely contained in the image of f. Consider next the composite

$$g\colon I^n \xrightarrow{\approx} D^n \to X^{(n-1)} \cup_a e^n \xrightarrow{f} Z$$

We are in good position to apply Lemma 2.1 and find a map $g_1 \simeq g$ relative to $g^{-1}(W)$ which is PL on a polyhedron $K \subset I^n$ whose image lies in $\mathring{e}^k$. Since g sends the boundary of the cube to W, it has not moved during the (relative) homotopy, and we can extend this homotopy constantly outside so as to get a map $f_1 \simeq f$ with $f_1 \mid_{e^n}$ is given by g. Since the dimension of the (compact) polyhedron K is strictly smaller than k, its image $f_1(K)$ only hits a finite number of affine subspaces of $\mathbb{R}^k \approx \mathring{e}^k$ in a neighborhood we called U in the lemma. Therefore there is a point u in U which does not belong to the image of f_1.

We have thus managed to be in the situation where the map f hits the cell e^k but misses an interior point. The space $W \cup (e^k \setminus u)$ admits a deformation retraction down to W since a pricked disc retracts to its boundary sphere. This yields a map $f_2 \simeq f_1 \simeq f$ which misses the cell e^k. If needed, we repeat this argument, finitely many times, for all large cells in the image of f, obtaining in the end a homotopic map which is cellular.

But this map has only been constructed for one extra cell in X. In general $X^{(n)}$ has been constructed from the $(n-1)$-skeleton by attaching many cells, maybe infintely many. We apply the same procedure simultaneously to all cells, so as to get a map f_n defined on $X^{(n)}$, we extend the homotopy to a homotopy defined on the whole space X just as we did for the 0-cells in the first step of this proof.

To conclude we need to assemble the successive homotopies we have constructed skeleta by skeleta. Because X might be infinite dimensional, we concatenate possibly an infinite number of homotopies. To do so let us spend half a second to perform the homotopy yielding a map which is cellular on the 0-skeleton, then one quarter of a second to have it cellular on the 1-sekelton, one eighth of a second to continue to the 2-sekeleton, etc. This defines a homotopy from f to f_∞ which is is cellular on X. □

As promised in Example 1.2 we cash in right away the benefits for our hard work.

Corollary 2.3. *For any* $0 \leq n < k$ *we have* $\pi_n S^k = 0$.

Proof. Any map $f\colon S^n \to S^k$ is homotopic to a cellular map, but the latter is constant. □

This property that all lower homotopy groups are trivial is important enough to deserve a name.

Definition 2.4. A space X is *n-connected* if $\pi_k X = 0$ for all $k \leq n$ and any choice of base point. A pair (X, A) is n-connected if $\pi_k(X, A) = 0$ for all $k \leq n$ and any choice of base point.

Example 2.5. A space X is 0-connected if $\pi_0 X$ is reduced to a point, i.e. X is path-connected. A space is 1-connected if moreoever $\pi_1 X = 1$, i.e. it is simply connected. We have seen that the n-sphere is $(n-1)$-connected and we will see that more generally spaces built from large cells are highly connected, more precisely, any CW-complex whose n-skeleton is reduced to a point is n-connected.

Remark 2.6. For a pair (X, A) the long exact sequence in homotopy tells us that being n-connected means that the first homotopy groups of A and X agree: $\pi_k A \cong \pi_k X$ for $k < n$ and the next one $\pi_n A \to \pi_n X$ is an epimorphism.

3. CW-approximation

In the previous section we proved that any map between CW-complexes can be chosen to be cellular, up to homotopy. In this section we deal with arbitrary spaces and show that one can replace them with CW-complexes, up to weak equivalence.

Definition 3.1. An unpointed map $f\colon X \to Y$ is a *weak homotopy equivalence* if it induces isomorphisms $f_*\colon \pi_n(X; x_0) \to \pi_n(Y; f(x_0))$ for all $x_0 \in X$.

This means that X and Y have the same number of path-connected components and on each of them f induces an isomorphism on all homotopy groups. We have met examples of weak (homotopy) equivalences which are not homotopy equivalences, such as the inclusion of a point in the Warsaw circle, but of course homotopy equivalences are weak equivalences since an inverse up to homotopy induces an inverse isomorphism on homotopy classes.

Before getting to the proof of the CW-approximation Theorem, we state and prove two small lemmas about certain highly connected pairs appearing in our constructions.

LEMMA 3.2. *The pair $(X \vee S^n, X)$ is $(n-1)$-connected and the inclusion $i\colon X \hookrightarrow X \vee S^n$ induces an isomorphism on π_{n-1}.*

PROOF. The wedge summand inclusion $X \hookrightarrow X \vee S^n$ admits a retraction, namely the collapse of the sphere. Therefore i_* is injective in any degree. Moreover, when $k < n$, the Cellular Approximation Theorem 2.2 (or rather its relative version for pairs) tells us that any map $(D^k, S^{k-1}) \to (X \vee S^n, X)$ factors through the pair (X, X) up to homotopy when $k < n$. As $\pi_k(X, X) = 0$, this shows the triviality of the relative homotopy groups in degrees $k < n$ and the surjectivity of i_* by inspection of the long exact sequence in homotopy for the pair $(X \vee S^n, X)$. □

In the above lemma we could do a little better than for an arbitrary attaching map because the n-cells we added were attached with a trivial attaching map, yielding a wedge. In general we can say the following.

LEMMA 3.3. *Let $a\colon S^n \to X$ be any map. The pair $(X \cup_a e^{n+1}, X)$ is n-connected. In particular the inclusion $X \hookrightarrow X \cup_a e^{n+1}$ induces an isomorphism on all π_k for $k < n$.*

PROOF. The triviality of $\pi_k(X \cup_a e^{n+1}, X)$ for $k \leq n$ is provided as in the previous proof by the Cellular Approximation Theorem (for pairs). The long exact sequence in homotopy allows us to conclude. □

THEOREM 3.4. *Any space X admits a* CW-approximation*, i.e. a weak equivalence $f\colon Z \to X$ from a CW-complex Z.*

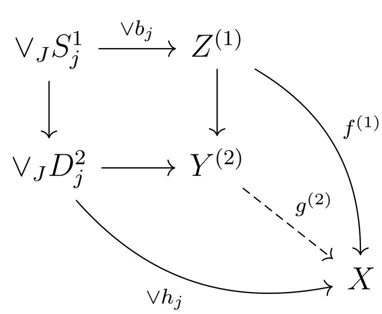
Proof. We assume that X is path connected and otherwise we apply the following construction on each path-connected component. The construction is an inductive process on the dimension of the cells.

We choose $Z^{(0)} = *$ to be a point and define $f^{(0)}\colon * \to X$ by sending this point to our favorite point $x_0 \in X$. This induces a bijection on π_0 by assumption. The construction to obtain an isomorphism on the next homotopy group, is done in two steps. We first obtain a surjection and then correct it by killing the kernel.

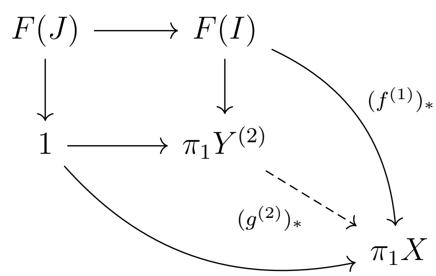
Let us attach 1-cells to our point $*$ so as to construct $Z^{(1)}$ which is a wedge of circles $\vee_{i\in I} S^1_i$. Here I is a set in bijection with a choice of generators α_i of $\pi_1(X; x_0)$. We represent the α_i's by pointed maps $a_i\colon S^1 \to X$ and assemble them to define $f^{(1)}\colon Z^{(1)} \to X$ (the restriction to S^1_i is a_i). On fundamental groups this induces a homomorphism $\pi_1(\vee S^1) \cong F(I) \to \pi_1 X$. The fundamental group of a wedge of circles is a free group whose generators we call x_i, for $i \in I$, by the Seifert-van Kampen Theorem, and by construction x_i is sent to α_i. In particular this homomorphism is surjective and we call the kernel $K = \mathrm{Ker}(f^{(1)})_*$.

In our second step we choose generators β_j of this kernel, where $j \in J$ and quickly represent them by maps $b_j\colon S^1_j \to Z^{(1)} = \vee_I S^1)$. By definition of K they have the property that $f^{(1)} \circ b_j$ are all null-homotopic. Choose a null-homotopy B_j defined on $S^1_j \times I$ with $B_j(s,1) = *$ for all $s \in S^1_j$. By the universal property of the quotient, the homotopy B_j induces a map on the space obtained by collapsing the top lid $S^1_j \times 1$. Let us call $h_j\colon D^2_j \approx (S^1_j \times I)/(S^1_j \times 1) \to X$.

Let us define $Y^{(2)} = X^{(1)} \cup (\vee D^2_j)$ where the j-th 2-cell is attached via B_j. This allows us to complete the following diagram with the dotted arrow since the interior square is a pushout square by construction and the bended arrows make the outer square commute:



We show now that our new map $g^{(2)}$ induces an isomorphism on the fundamental group. Apply the functor π_1 to the whole diagram and identify with the corresponding free groups the fundamental groups of wedges of spheres:



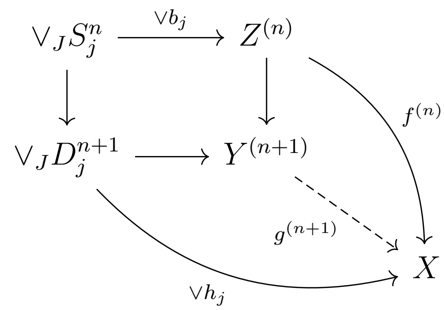
By the Seifert-van Kampen Theorem, the inner square is a pushout diagram of groups and $(g^{(2)})_*$ is an isomorphism. We continue by induction adding 2-cells to obtain a surjection on π_2 and rectifying this by adding 3-cells, etc.

So, let us assume that we have constructed an n-dimensional CW-complex $Y^{(n)}$ and a map $g^{(n)}\colon Y^{(n)} \to X$ inducing an isomorphism on homotopy groups π_k for $k < n$. Just as before we will first add n-cells to obtain a surjection on π_n and then rectify it to get an isomorphism. One reason we went through the argument for π_1 separately is that we had the Seifert-van Kampen to perform the explicit computation of the fundamental group, but there is no higher version we can use now.

Using abusive notation, let us call (again) α_i chosen generators for $\pi_n X$ and $a_i\colon S^n \to X$ some representatives, for $i \in I$. Form a wedge of spheres $\bigvee_I S^n_i$ and define $Z^{(n)} = Y^{(n)} \vee \bigvee_I S^n$. We deduce from Lemma 3.2 that the inclusion $Y^{(n)} \hookrightarrow Z^{(n)}$ induces an isomorphism on π_k for all $k < n$. We then extend $g^{(n)}$ to a map $f^{(n)}\colon Z^{(n)} \to X$ by using a_i on S^n_i. This induces a surjection on π_n since composite maps

$$S^n \xrightarrow{\iota_i} \vee_J S^n_i \hookrightarrow Y^{(n)} \vee \bigvee_J S^n \xrightarrow{f^{n)}} X$$

is equal to a_i. In order to construct $Y^{(n+1)}$ we choose generators β_j of the kernel K of $(f^{(n)})_*\colon \pi_n Z^{(n)} \to \pi_n X$ and represent them by maps $b_j\colon S^n \to Z^{(n)}$. Just as above we choose null-homotopies $h_j\colon D^{n+1}_j \to X$ and define $Y^{(n+1)}$ to be the pushout in the small square below



The dotted map $g^{(n+1)}$ is induced by the universal property of the pushout. By Lemma 3.3 we know that the pair $(Y^{(n+1)}, Z^{(n)})$ is n-connected. All together, these considerations about the connectivity of the pairs imply that $\pi_k Y^{(n)} \to \pi_k Z^{(n)} \to \pi_k Y^{(n+1)}$ are isomorphisms for all $k < n$. As we assumed that these homotopy groups agreed with those of X we conclude that $g^{(n+1)}$ induces isomorphism $\pi_k Y^{(n+1)} \to \pi_k X$ for all $k < n$.

Let us finally look at the effect of $g^{(n+1)}$ on π_n. The easy part is surjectivity since $(f^{(n)})_*\colon \pi_n Z^{(n)} \to \pi_n X$ is so by construction and this surjection factors through $(g^{(n+1)})_*$. Injectivity follows from a less direct argument. Let κ belong to the kernel of $(g^{(n+1)})_*$ and represent it by a map $k\colon S^n \to Y^{(n+1)}$. We have seen that the pair $(Y^{(n+1)}, Z^{(n)})$ is n-connected, so k lifts up to homotopy to a map $k'\colon S^n \to Z^{(n)}$. Therefore $(g^{(n+1)})_*(\kappa)$ is represented by the composite map $f^{(n)} \circ k'$. We picked κ in the kernel, so $\kappa' = [k']$ belongs to the kernel K.

This kernel is an abelian group (being a subgroup of a higher homotopy group), it is thus a finite sum of generators β_j or their opposites $-\beta_j$. This means that up to homotopy k' factors through the wedge $\vee_J S^n_j$: if we need to introduce ℓ times β_j for $\ell \in \mathbb{N}$, we pinch the corresponding sphere ℓ times and use $p\colon S^n \to \vee_\ell S^n$, and if ℓ is negative we precompose with the degree -1 map (changing the sign of one coordinate for example). This shows, looking at the pushout diagram above, that k factors through the wedge of discs, which is contractible. Thus k is null-homotopic and we are done.

The CW-approximation is completed by setting $Z = \cup Z^{(n)}$ and using the compatible maps $f^{(n)}$ to define f on Z. It induces an isomorphism on all homotopy groups. Focusing on one of them, π_n say, we have shown that $f^{(n+1)}$ induces an isomorphism on π_n and from there on, attaching higher cells does not change this homotopy group. □

Remark 3.5. There is a more general version of the CW-approximation Theorem, relative to a subspace $A \subset X$ to which one attaches cells so as to construct as relative CW-complex (Z, A), weakly equivalent to the pair (X, A).

All together what we have seen in the first sections of this chapter is that, up to weak homotopy equivalence, one can replace any space by a CW-complex, and then any map between such nice spaces by a cellular map. This retricts our study to a manageable class of spaces and maps where the methods of the Algebraic Topology course apply nicely.

What we have not done yet (and promise to come back to later) is to show that the CW-approximation of a space is unique up to weak homotopy equivalence. We will need mapping cylinders for that. Then we will show that in fact a weak equivalence between CW-complexes is a homotopy equivalence.

4. Postnikov sections

The technique we have seen in Section 3 will be applied next not to rebuild the correct homotopy groups of a CW-approximation, but to kill all higher homotopy groups, thus obtaining more complicated spaces from the cellular construction, but simpler from the point of view of homotopy groups. We will actually construct a tower of spaces $X[n]$ living under X (together with maps $X \to X[n]$) and differing from one to the next in a single homotopy group. Spaces with a single non-trivial homotopy group are important enough to get a name, or rather two.

Definition 4.1. Let A be a group and $n \geq 1$. A path-connected space X such that $\pi_k X = 0$ for all $k \neq n$ and $\pi_n X \cong A$ is called an *Eilenberg-Mac Lane space* of type $K(A, n)$.

For now, as we do not require a $K(A, n)$ to be a CW-complex, there is no reason why such Eilenberg-Mac Lane spaces should be unique up to homotopy.

Example 4.2. We have already met a few Eilenberg-Mac Lane spaces. When $n = 1$ they correspond to spaces with contractible universal cover:

(1) The circle S^1 is a $K(\mathbb{Z}, 1)$.
(2) The torus $S^1 \times S^1$ is a $K(\mathbb{Z} \times \mathbb{Z}, 1)$.
(3) The wedge $S^1 \vee S^1$ is a $K(\mathbb{Z} * \mathbb{Z}, 1)$.
(4) The infinite real projective space $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2, 1)$.
(5) The infinite complex projective space $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$, but this is harder to prove...

We start right away with the construction of *Postnikov sections*, and as a particular case we will be able to construct $K(A, n)$'s for all n and all groups A (abelian when $n \geq 2$).

PROPOSITION 4.3. *Let X be a path-connected space and $n \geq 1$. There exists a space $X[n]$ and a map $\ell_n\colon X \to X[n]$ such that $(l_n)_*\colon \pi_k X \to \pi_k X[n]$ is an isomorphism for all $k \leq n$ and $\pi_k X[n] = 0$ for $k > n$.*

PROOF. We choose a set of generators $\alpha_i \in \pi_{n+1} X$ and represent them by maps $a_i\colon S^{n+1} \to X$. We then construct the pushout

$$\begin{array}{ccc} \vee_I S_i^{n+1} & \xrightarrow{\vee a_i} & X \\ \downarrow & & \downarrow \\ \vee_I D_i^{n+2} & \longrightarrow & X' \end{array}$$

We know from Lemma 3.3 that the pair (X', X) is $(n+1)$-connected as we attach cells of dimension $n+2$. This implies that the inclusion induces isomorphisms $\pi_k X \cong \pi_k X'$ for $k \leq n$ and an epimorphism on π_{n+1}. But any map $S^{n+1} \to X'$ factors through its $(n+1)$-st skeleton $(X')^{(n+1)} = X^{(n+1)}$ up to homotopy, so its homotopy class comes from a class in $\pi_n X$. These classes become null-homotopic by construction, so $\pi_{n+1} X' = 0$.

We iterate this construction and kill $\pi_{n+2} X'$ by constructing a space X'' from X' by attaching $(n+3)$-cells. The union of these spaces $X \subset X' \subset X'' \subset \dots$ is called $X[n]$ and enjoys the desired properties. □

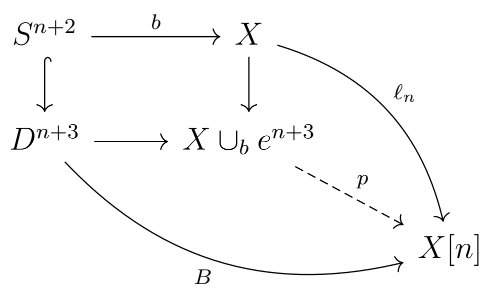
With this construction $X[0]$ is a weakly contractible space (all its homotopy groups are trivial), and $X[1]$ is a $K(\pi_1 X, 1)$.

We can assemble all Postnikov sections into a tower.

THEOREM 4.4. *Let X be a path-connected space. There is a tower of maps* $\cdots \to X[n+1] \xrightarrow{p_n} X[n] \to \cdots \to X[1] \to X[0]$ *such that* $p_n \circ \ell_n = \ell_{n+1}$.

PROOF. In order to construct the map $p_n\colon X[n+1] \to X[n]$, we use the construction from the proof of Proposition 4.3. Our aim is to extend the inclusion $X \subset X[n]$ to a map on $X[n+1]$. The latter space has been constructed by attaching cells of dimension $\geq n+3$, the lowest dimensional ones having been used to kill $\pi_{n+2}X$.

Instead of doing the full inductive argument on all cells, let us only look at one of these cells e^{n+3} and its attaching map b. We have a pushout square as below:



where the map B is a nullhomotopy for $\ell_n \circ b$, which exists since $\pi_{n+2}X[n] = 0$. The diagram therefore commutes and the universal property of the pushout provides the dashed arrow p. We could have done that for a wedge of spheres, and then iterate so as to construct p_n. □

EXAMPLE 4.5. Since S^2 is simply-connected, we have a weakly contractible first Postnikov section $S^2[1]$. We also know that $\pi_2 S^2 \cong \mathbb{Z}$, so $S^2[2]$ is a $K(\mathbb{Z}, 2)$. The next Postnikov section is more interesting. The Hof map generates $\pi_3 S^2 \cong \mathbb{Z}$, so $S^2[3]$ is a space with two non-trivial homotopy groups, in degree 2 and 3, both being isomorphic to $\mathbb{Z}$. We will see that this space is not a product $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3)$, the Eilenberg-Mac Lane spaces are glued together in a "twisted way".

CHAPTER 4

Fibrations and cofibrations

In the previous chapter we have given concrete constructions that allow us to work with nice spaces (CW-complexes) and nice maps (cellular maps). Now we take some time to study a formal setup in which it is convenient to do homotopy theory. Two classes of maps play a central role in this theory, hinting at the structure of a model category, a notion due to Quillen, [**7**]. This notion is one of the main topics covered in the course Homotopical Algebra, so one of our objectives will be to present one way to do homotopy theory with spaces in a way that can be seen as a guideline to generalizations.

We will also study long exact sequences associated to fibrations and cofibrations.

1. Mapping cylinders and mapping cones

We follow Sections 1 and 6 from [**12**, Chapter 4]. The mapping cylinder construction will be very useful to "turn a map into a cofibration". We have not defined yet what it means to be a cofibration, but let us think about a nice subspace inclusion, like a sub-CW-complex.

Definition 1.1. . Let $f\colon X \to Y$ be a map. The *mapping cylinder* $\mathrm{Cyl}(f)$ is the space $(X \times I) \coprod Y/(x, 0) \sim f(x)$.

Remark 1.2. The cylinder construction actually defines a functor on the category of maps (morphisms are commutative squares). If the map is pointed and we wish to stay in the pointed category, then we would use the pointed version of the cylinder $X \rtimes I$ so the cylinder has a canonical base point. We will not systematically develop the whole theory in both settings, but it is usually quite obvious to adapt the unpointed version to the pointed one. In this chapter we will concentrate in fact on the pointed version since we are mostly interested in pointed homotopy classes of maps.

LEMMA 1.3. *Let $f\colon X \to Y$ be a map. The collapse map $r\colon \mathrm{Cyl}(f) \to Y$ is a homotopy equivalence.*

PROOF. The collapse map is defined by $r(x,t) = \overline{(x,0)} = \overline{f(x)}$ for all $x \in X, t \in I$, and $r(y) = \overline{y}$ for all $y \in Y$. We have already seen this argument in the topology course: The inclusion $i\colon Y \hookrightarrow \mathrm{Cyl}(f)$ has r as a strong deformation retract. Obviously $r \circ i$ is the identity and $i \circ r$ is homotopic to the identity relative to Y via the homotopy

$$H\colon \mathrm{Cyl}(f) \times I \longrightarrow \mathrm{Cyl}(f)$$
$$(\overline{(x,t)}, s) \longmapsto \overline{(x, st)}$$
$$(y,s) \longmapsto y$$

This homotopy collapses slowly the cylinder on X down to its base in 1 second. □

We define now the mapping cone of a map f as above as the target space to which we attach a cone on X. We have already met this kind of spaces in topology to attach what we called X-cells.

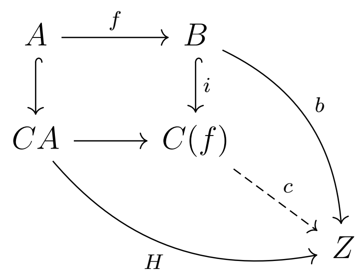
DEFINITION 1.4. Let $f\colon X \to Y$ be a map. The *mapping cone* $C(f)$ is the quotient space $\mathrm{Cyl}(f)/X \times 1$.

In topology the sequence $X \xrightarrow{f} Y \to C(f)$ behaves really like an exact sequence. Let us thus introduce the terminology and prove this in the next lemma.

DEFINITION 1.5. A sequence of pointed spaces $A \xrightarrow{f} B \xrightarrow{g} C$ is *h-coexact* if for any pointed space Z the sequence $[C,Z]_* \xrightarrow{g^*} [B,Z]_* \xrightarrow{f^*} [A,Z]_*$ is exact in pointed sets, i.e. $(f^*)^{-1}[c_{z_0}] = \mathrm{Im} g^*$.

LEMMA 1.6. *The sequence $A \xrightarrow{f} B \xrightarrow{i} C(f)$ is h-coexact.*

PROOF. Since $i \circ f$ is the inclusion of A at the bottom of the cone on A, this composition is null-homotopic, so $f^* \circ i^*$ is constant. Conversely, by definition of the cone of a map, we have a pushout square defining $C(f)$ and we add to the picture a map $b\colon B \to Z$ such that $f^*[b] = [c_{z_0}]$:



Since $b \circ f$ is homotopic to the constant map, there is a null-homotopy H defined on the cone of A. It deforms continuously $b \circ f$ at the bottom of the cone to the constant map defined on the top of the cone, which means that the diagram above commutes. Therefore the dashed arrow c exists, making the whole diagram commute. In particular $c \circ i = b$, hence $i^*[c] = [b]$. $\square$

Now that we have started constructing an h-coexact sequence from any map, we can iterate the construction. So we wish to construct the cone of the inclusion $f_1\colon B \hookrightarrow C(f)$, i.e. take $C(f)$ and attach a cone on B. This would be nice maybe, but not so interesting if we were not able to identify the homotopy type of $C(i)$ in terms of the previous data.

LEMMA 1.7. *The cone $C(f_1)$ is homotopy equivalent to the suspension ΣA.*

PROOF. The idea is that the cone of B we attach to $C(f_1)$ is even larger than the second cone on A we would attach to CA to construct the suspension. More precisely, let us look at the following commutative diagram starting with two pushout squares on the left and completing then with quotient maps on the right:

$$\begin{array}{ccccc}
A & \hookrightarrow & CA & \longrightarrow & CA/A = \Sigma A \\
\downarrow f & & \downarrow & & \downarrow \approx \\
B & \overset{f_1}{\hookrightarrow} & C(f) & \overset{p}{\longrightarrow} & C(f)/B \\
\downarrow & & \downarrow & & \downarrow \approx \\
CB & \hookrightarrow & C(f_1) & \overset{q}{\longrightarrow} & C(f_1)/CB
\end{array}$$

where p and q denote the quotient maps. Since we started with pushout squares, the right hand side vertical maps are homeomorphisms, so in particular the induced map $\Sigma A \to C(f_1)/B$ is a homeomorphism. To conclude we show that the map q is a homotopy equivalence. For the sake of readability, we will not write bars over the pairs of the form (x, s) representing elements in cones and suspensions.

Define $s(f)\colon \Sigma A \to C(f_1)$ by sending the upper cone C_+A to CA via $s(f)(a, s) = (a, 2s)$ and the lower cone C_-A to CB via $s(f)(a, s) = (f(a), 2(1-s))$. When $s = 1/2$, we send $(a, 1/2)$ to the base of the cone CA, that is on $(a, 1)$ (the top of the cone is the class of $(a, 0)$), and this is identified with $(f(a), 1)$ in the base of the cone on B.

Let us write down now an explicit homotopy contracting CB to a point.

$$\begin{aligned} H\colon C(f_1) \times I &\longrightarrow C(f_1) \\ (a, s, t) &\longmapsto \begin{cases} (a, (1+t)s) & \text{if } (1+t)s \leq 1 \\ (f(a), 2-(1+t)s) & \text{else} \end{cases} \\ (b, s, t) &\longmapsto (b, (1-t)s) \end{aligned}$$

To check that this piecewise formula gives a continuous map, we just have to check that if $(1+t)s = 1$, both formulas $(a, (1+t)s) = (a, 1)$ and $(f(a), 2-(1+t)s) = (f(a), 1)$ define the same element, which is the case by definition of $C(f_1)$, and moreover two elements (a, s, t) and (b, s, t) in the same class have the same image under H. This happens for $b = f(a)$ and $s = 1$, where both formulas give $(f(a), 1-t)$. So continuity is established.

We check now what happens at $t = 0$ and $t = 1$. When $t = 0$, we have $H(a, s, 0) = (a, s)$ and $H(b, s, 0) = (b, s)$, this is the identity. At $t = 1$ we get on the one hand $H(b, s, 1) = (b, 0)$, so $H(-, 1)$ is the constant map on the cone on B. On the other hand

$$H(a, s, 1) \longmapsto \begin{cases} (a, 2s) & \text{if } 2s \leq 1 \\ (f(a), 2-2s) & \text{else} \end{cases}$$

This is exactly $s(f)$. In other words, $H(-, 1) = s(f) \circ q$, we first collapse the cone on B to a point and continue with $s(f)$. This shows that $s(f) \circ q$ is homotopic to the identity on $C(f_1)$.

To conclude we still need to check the other composition, namely $q \circ s(f)$. When $s > 1/2$, we see that $s(f)(a, s) = (f(a), 2(1 - s))$ is an element in the cone on B, which is collapsed by q, so the lower cone $C_{-}A$ is sent to the base point of ΣA. On the upper cone, $s \geq 1/2$, we have

$$(q \circ s(f))(a, s) = q(a, 2s) = (a, 2s)$$

This composition simply collapses the lower cone. This is also homotopic to the identity, a homotopy is given by collapsing only the part $A \times [0, t]$ of the lower cone $(A \times [0, 1])/(A \times 0)$. $\square$

Before continuing even one step further, let us look again at the h-coexact sequence we constructed from f, and completed by constructing the cone on f_1:

$$A \xrightarrow{f} B \xrightarrow{f_1} C(f) \xrightarrow{f_2} C(f_1)$$

From the previous lemma we learn that we can replace $C(f_1)$ by ΣA and the map f_2 by the simpler map $p\colon C(f) \to \Sigma A$ collapsing the bottom cone CB. After this slow start, let us continue and identify $C(p)$! We will see a minus sign appearing in a map between suspensions: this is given by precomposing with ι, the inverse for the co-H-group structure we have seen in 2.10.

Proposition 1.8. *The space $C(p)$ is homotopic to ΣB and the inclusion map $\Sigma A \hookrightarrow C(p)$ is then replaced $-\Sigma f\colon \Sigma A \to \Sigma B$.*

Proof. The first claim about the homotopy type of $C(p)$ follows directly from Lemma 1.7, the surprising part is maybe the minus sign appearing when one identifies the map $\Sigma A \to \Sigma B$. Just as we have replaced $C(f_1)$ by ΣA via the collapse map q, we replace $C(f_2)$ by ΣB via a collapse map q'. More explicitly this means that in $C(f_2) = C(f_1) \cup C(C(f))$ we collapse the whole cone $C(C(f))$:

$$\begin{array}{ccccccc}
B & \longrightarrow & C(f) & \xrightarrow{f_2} & C(f_1) & \xrightarrow{f_3} & C(f_2) \\
 & & & \searrow^{p} & \downarrow q & \searrow^{p'} & \downarrow q' \\
 & & & & \Sigma A & \xrightarrow{-\Sigma f} & \Sigma B
\end{array}$$

We complete the diagram with the diagonal map $p' = q' \circ f_3$. The second claim will then follow if we prove that the middle triangle commutes up to homotopy. Instead of proving that directly, we precompose with the map $s(f)\colon \Sigma A \to C(f_1)$ constructed in the previous proof. We have seen that $q \circ s(f)$ is homotopic to the identity, so we wish to show that $p' \circ s(f)$ is homotopic to $-\Sigma f$. Let us compute this composition.

For an element in the upper cone C_+A, the image $s(f)(a,s) = (a,2s)$ belongs to the cone on $C(f)$ so it is sent to the base point under the collapse map p'. For an element on the lower cone C_-A, we have $p'[s(f)(a,s)] = p'(f(a), 2(1-s))$. This goes twice as fast, but is homotopic to the map $(a,s) \mapsto (f(a), 1-s)$, which is nothing but $\Sigma f \circ \iota$ since we reversed the orientation of the cylinder. □

We finally iterate one more time to get back the suspension of $C(f)$. To sum up we can splice up h-coexact sequences so as to get a long *Puppe sequence*.

THEOREM 1.9. *Let $f\colon A \to B$ be a pointed map. The sequence*

$$A \xrightarrow{f} B \xrightarrow{f_1} C(f) \xrightarrow{p} \Sigma A \xrightarrow{-\Sigma f} \Sigma B \xrightarrow{-\Sigma f_1} \Sigma C(f) \xrightarrow{-\Sigma p} \Sigma^2 A \xrightarrow{\Sigma^2 f} \dots$$

is then h-coexact.

PROOF. We only need to observe that $\iota \circ \iota$ is homotopic to the identity to identify the second iteration of maps between double suspensions. □

REMARK 1.10. Taking $[-, Z]_*$ for any space Z yields a long exact sequence of pointed sets. As soon as we are deakling with suspensions, we know that this is an exact sequence of groups, and starting from the sixth term, we are looking at abelian groups. In fact one can saya little more at the place where we move from sets to groups. The pinch map $\mu\colon C(f) \to \Sigma A \vee C(f)$ pinching the copy $A \times 1/2$ at half height on the cone on A provides a map

$$[\Sigma A, Z]_* \times [C(f), Z]_* \cong [\Sigma A \vee C(f), Z]_* \to [C(f), Z]_*$$

which is a group action. Two elements in $[C(f), Z]_*$ have the same image in $[B, Z]_*$ if and only if they belong to the same orbit under the action of the group $[\Sigma A, Z]_*$.

2. Path spaces and loop spaces

In this section we dualize the theory and construct a certain h-exact sequence. Cones will be replaced by path spaces, suspensions by loop spaces, the rest is formal.

DEFINITION 2.1. A sequence of pointed spaces $X \xrightarrow{f} Y \xrightarrow{g} Z$ is *h-exact* if for any pointed space A the sequence $[A, X]_* \xrightarrow{f_*} [A, Y]_* \xrightarrow{g_*} [A, Z]_*$ is exact in pointed sets, i.e. $(g_*)^{-1}[c_{z_0}] = \mathrm{Im} f_*$.

As indicated above the role of the cone is played by a path space.

DEFINITION 2.2. Let X be a pointed space. The *path space* FX is the space $\mathrm{map}_*(I, X)$ where 0 is the base point of the interval I.

Just like A is nicely contained in the cone CA as a subspace, and this cone is a contractible space, we have dually a "nice" surjection $ev_1 \colon FX \to X$ from a contractible path space (one can contract every path down to the base point). This duality is also illustrated by the fact that a map $f \colon A \to X$ is nullhomotopic if and only if it admits an *extension* to the cone $A \subset CA$, or dually if and only if it admits a *lift* to $FX \to X$. This is immediate by adjunction, but is very useful to have in mind.

DEFINITION 2.3. Let $f \colon X \to Y$ be a pointed map. The *mapping fiber* $F(f)$ is the pullback of the diagram $X \xrightarrow{f} Y \xleftarrow{ev_1} F(Y)$. We write $f^1 \colon F(f) \to X$ for the map provided by this pullback construction.

Hence, points of $F(f)$ are pairs (x, ω) consisting of a point $x \in X$ and a path $\omega \colon I \to Y$ starting at y_0 such that $\omega(1) = x$.

We will now only state the dual statements to those from the previous section. The proofs are ... dual.

LEMMA 2.4. *The sequence* $F(f) \xrightarrow{f^1} X \xrightarrow{f} B$ *is h-exact.*

PROPOSITION 2.5. *The space* $F(f^1)$ *is homotopic to* ΩY*, the space* $F(f^2)$ *is homotopic to* ΩX *and the map* f^3 *is homotopic to* $-\Omega f \colon \Omega X \to \Omega Y$.

THEOREM 2.6. *Let $f\colon X \to Y$ be a pointed map. The sequence*

$$\cdots \to \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{-\Omega p} \Omega F(f) \xrightarrow{-\Omega q} \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{p} F(f) \xrightarrow{q} X \xrightarrow{f} Y$$

is then h-exact.

EXAMPLE 2.7. An h-exact sequence yields an exact sequence of pointed sets of homotopy classes for any pointed space used as a source. A particularly interesting choice is S^0 since $[S^0, \Omega^n X]_* \cong [S^n, X]_* = \pi_n X$ by adjunction. We thus get a long exact of homotopy groups

$$\cdots \to \pi_2 X \to \pi_2 Y \to \pi_1 F(f) \to \pi_1 X \to \pi_1 Y \to \pi_0 F(f) \to \pi_0 X \to \pi_0 Y.$$

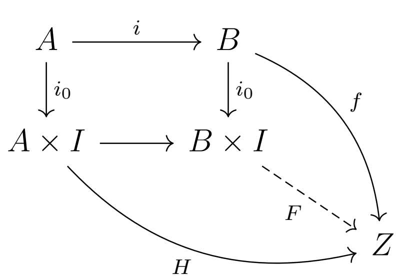
This time we have not bothered to indicate the relevant minus signs since image and kernel of a map or its opposite are equal.

To conclude this short section, let us try to understand what the homotopy groups of the mapping fiber represent. The elements of $\pi_n F(f)$ are homotopy classes of pointed maps $S^n \to F(f)$ and since $F(f)$ is defined as apullback, they correspond to compatible pairs of maps $S^n \to X$ and $S^n \to F(Y)$. The adjoint of the latter is a map $S^n \wedge I \approx D^{n+1} \to Y$, i.e. a homotopy from $f \circ \alpha$ to a constant map. In other words, when f is a subspace inclusion $X \subset Y$, then we are looking at $\pi_{n+1}(Y, X)$.

3. The homotopy extension property

Now that we have seen how gluing cones, or dually assembling mapping fibers, yields "exact sequences up to homotopy", we are ready to introduce the homotopy extension property that lies at the heart of the notion of cofibration.

DEFINITION 3.1. A map $i\colon A \to B$ has the *homotopy extension property*, or HEP for short, with respect to a space Z if for each solid arrow commutative diagram



there exists a homotopy F extending H and starting at f.

In other words, we are given a homotopy H on A and know how to extend the map $H(-,0)$ to B. We wish to extend this homotopy H to a space B, which we think of as a larger space containing A.

DEFINITION 3.2. A map $i\colon A \hookrightarrow B$ is a *cofibration* if it has the homotopy extension property with respect to all spaces.

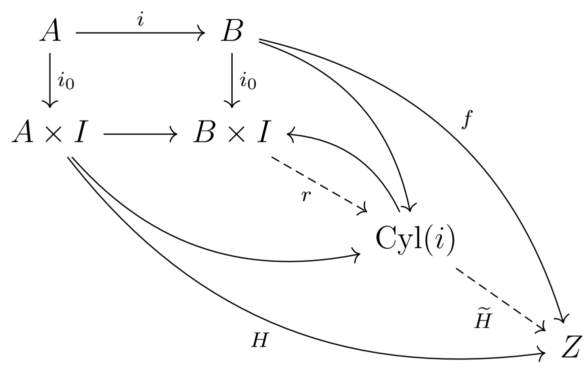
EXAMPLE 3.3. The inclusion $i\colon S^{n-1} \subset D^n$ is a cofibration. This is the argument we have already used in the Cellular Approximation Theorem 2.2. The extension problem we have to solve is described in Definition 3.1. With the same notation let us thus assume that we have a map $f\colon D^n \to Z$ and a homotopy $H\colon S^{n-1} \times I$ starting at $f\mid_{S^{n-1}}$. By the universal property of the pushout this means that f and H assemble to yield a map $\widetilde{H}$ defined on an "empty cylinder" $\mathrm{Cyl}(i) = (D^n \times 0) \cup (S^{n-1} \times I)$.

We claim this map extends to a map on the full cylinder $D^n \times I$ and the reason is that the inclusion $\mathrm{Cyl}(i) \subset D^n \times I$ admits a strong deformation retract r defined as follows. We choose a point, say $(O;2)$ where O is the center of the ball D^n and for each point $(x;t) \in D^n \times I$ we define $r(x;t)$ to be the intersection of the line passing through $(x;t)$ and $(O;2)$ with $\mathrm{Cyl}(i)$. The points on the empty cylinder are obviously fixed by r and the homotopy, which we will not really use in this argument, is provided by moving linearly on the segment between $(x;t)$ and its image $r(x;t)$ in one second. We simply choose $F = H \circ r$.

The argument in the previous example shows that in general the HEP only depends on the way the cylinder of $i\colon A \to B$ is included in $B \times I$.

LEMMA 3.4. *A map i is a cofibration if and only it has the HEP with respect to the cylinder* $\mathrm{Cyl}(i)$.

PROOF. The direct implication is obvious, so let us assume that i has the HEP with respect to $\mathrm{Cyl}(i)$ and let us solve an arbitrary homotopy extension problem as indicated in the following diagram where we have added the cylinder on i (it is important to remember that $\mathrm{Cyl}(i)$ is the pushout of the diagram $A \times I \xleftarrow{i_0} A \xrightarrow{i} B$):



The solid arrow square ending at the cylinder on i is the pushout square we have mentioned above, so the map $\widetilde{H}$ exists by the universal property. By assumption i enjoys the HEP with respect to $\mathrm{Cyl}(i)$ so r exists by definition of the HEP. We conclude by choosing $F = \widetilde{H} \circ r$ as our homotopy starting at f and extending H. □

The statement of the lemma could have been made even more precise since we have only used one specific lifting problem in the course of the proof, namely that the inclusion $\mathrm{Cyl}(i) \subset B \times I$ admits a retraction. We allow ourselves to talk about an inclusion here since cofibrations are necessarily embeddings, i.e. injective maps inducing a homeomorphism onto their image.

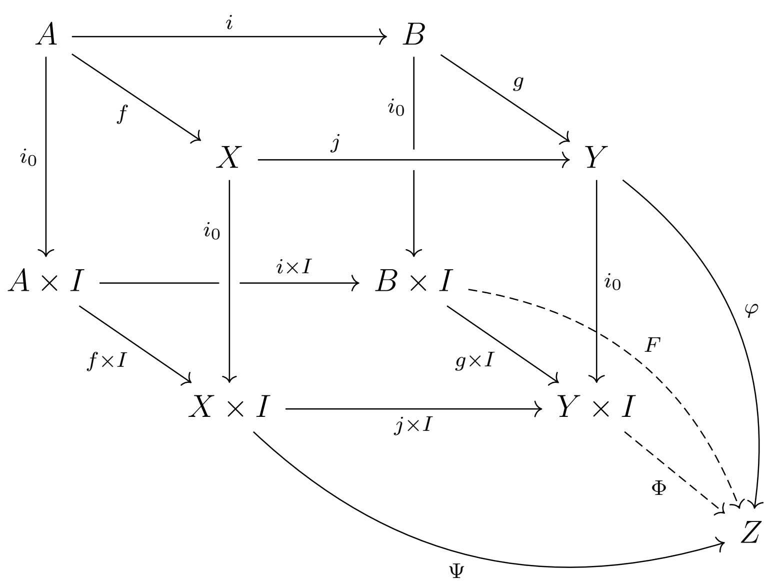
We continue with some basic properties of cofibrations, all of which will guide our intuition when (and if!) we will develop an abstract way of doing homotopy theory with so-called Quillen model categories.

Lemma 3.5. *An arbitrary coproduct of cofibrations is again a cofibration.*

Proof. The cylinder construction commutes with coproducts, being a left adjoint. Hence a homotopy extension problem for a coproduct is equivalent to a collection of homotopy extension problems. They all have a solution by assumption and we can use the coproduct of these homotopies to solve the original problem. □

Proposition 3.6. *The pushout of a cofibration along any map is a cofibration.*

PROOF. Consider a cofibration $i\colon A \to B$ and a map $f\colon A \to X$. We call Y the pushout and use the names of the natural maps as in the following diagram:



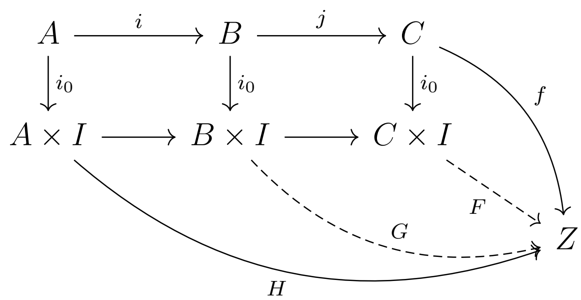
The map we claim is a cofibration is j and the homotopy lefting problem is given by the map φ and the homotopy Ψ. If $f = \varphi \circ g$ and $H = \Psi \circ (f \times I)$, then the associated homotopy lifting problem does admit a solution F since i is a cofibration.

Next, we use the fact that $- \times I$ preserves pushouts because it is a left adjoint, see Theorem 2.5. Thus, the universal property of the bottom pushout face of the cube provides a unique map $\Phi\colon Y \times I \to Z$. By construction this homotopy extends the homotopy Ψ and we only have to verify that it indeed starts at φ. This comes from the fact that both φ and $\Phi \circ i_0$ are maps out of Y that agree on B with $\varphi \circ g$ and on X with $\varphi \circ i$. We conclude by the universal property of the top pushout face of our cube. □

After having checked that cofibrations are stable under pushouts (sometimes called cobase change), we verify that they are also stable under composition.

LEMMA 3.7. *The composition of two cofibrations is a cofibration.*

Proof. Let $i\colon A \to B$ and $j\colon B \to C$ be two composable cofibrations. The result is transparent from the following diagram:



The solid arrow diagram shows which homotopy extension problem we wish to solve. The dashed arrow G exists by the HEP the first cofibration i enjoys, then we find F because j also is a cofibration. □

Proposition 3.8. *Let A be a filtered space with $\cup A_n = A$. If the inclusion $i_n\colon A_n \hookrightarrow A_{n+1}$ is a cofibration for any $n \geq 0$, then the inclusion $i\colon A_0 \to A$ is again a cofibration.*

Proof. Recall that $- \times I$ is a left adjoint, it therefore commutes with colimits. In particular $A \times I = \cup(A_n \times I)$. To solve a homotopy extension property for $A_0 \hookrightarrow A$ we solve it for $A_0 \hookrightarrow A_1$ and proceed inductively. □

After all these formal results we can use the elementary Example 3.3 to get a large class of very important cofibrations.

Theorem 3.9. *Let (X, A) be a relative CW-complex. The inclusion $A \subset X$ is then a cofibration. In particular the skeletal inclusions $X^{(n)} \subset X^{(n+1)}$ and $X^{(n)} \subset X$ are cofibrations for any CW-complex.*

Proof. A coproduct $\coprod S^{n-1} \hookrightarrow \coprod D^n$ of cofibrations is again a cofibration, therefore the process of attaching cells yields yet another cofibration $A \hookrightarrow A \cup (\bigcup e^n)$ by Proposition 3.6. We conclude by Proposition 3.8. □

A particular example of the above procedure is given by the inclusion of a point in a sphere $* \hookrightarrow S^n$ as this map is the pushout of our prototypical cofibration $S^{n-1} \hookrightarrow D^n$ along $D^n \to *$.

4. Turning maps into cofibrations and applications

Another important feature of cofibrations is that any map can be turned into such a nice inclusion, without changing the homotopy type of the target. With this trick in our pocket we will be ready to come back to the CW-approximation Theorem 3.4 and prove it is unique.

Let $f\colon X \to Y$ be any map. Without surprise the way to realize this is to use the cylinder $\mathrm{Cyl}(f)$.

Proposition 4.1. *Any map f admits a factorization $X \xrightarrow{i} \mathrm{Cyl}(f) \xrightarrow{p} Y$ into a cofibration i and a homotopy equivalence p.*

Proof. We have already seen that the map p, collapsing the cylinder $X \times I$ onto its base $X \times 0$ is a strong homotopy retraction of the inclusion $B \hookrightarrow \mathrm{Cyl}(f)$. The map i is given by $i(x) = (x, 1)$ and we prove it is a cofibration by applying Lemma 3.4. To construct a retraction $\mathrm{Cyl}(f) \times I \to \mathrm{Cyl}(i)$, we collapse first $Y \times I$ down to $Y \times 0$ and continue by working on $X \times I \times I$ which we project onto $(X \times I \times 0) \cup (X \times 1 \times I)$. Since the bottom part $X \times 0 \times I$ has been collapsed down to $X \times 0$ when we started with $Y \times I$, we have to project $X \times I \times 1$ in two parts. The bottom $X \times [0, 1/2] \times 1$ is dilated by a factor two and sent to $X \times I \times 0$ whereas the top part $X \times [1/2, 1] \times 1$ is also dilated by a factor two and sent to $X \times 1 \times I$. An explicit formula could be given, but a picture is probably more helpful. □

Our first application is to CW-approximation. We only state the absolute version (it says that two CW-approximations of the same space are weakly equivalent) but note, as with other results, that a relative version also holds.

Proposition 4.2. *Let $f\colon Z \to X$ and $f'\colon Z' \to X$ be two CW-approximations of the same pointed space X. There exists then a weak equivalence $h\colon Z \to Z'$ such that f and $h \circ f'$ are homotopic in the pointed category. This map h is unique up to pointed homotopy.*

Proof. Since CW-approximation is done path connected component by path connected component, we only deal with path connected spaces. We will also assume that Z is constructed by starting with a single 0-cell, this is how we have done it

Theorem 3.4. We turn f' into a cofibration as we have learned to do in Proposition 4.1 (using the pointed version of the cylinder $Z' \rtimes I$). Consider the diagram

$$\begin{array}{ccccc} Z & \xrightarrow{f} & X & & \\ & & \downarrow{\scriptstyle j} & & \\ Z' & \xrightarrow{i} & \mathrm{Cyl}(f') & \xrightarrow{r} & X \end{array} \qquad X \xrightarrow{id} X$$

Since $Z^{(0)}$ is a point and f is a pointed map we can consider $j \circ f$ as a map of pairs $(Z, Z^{(0)}) \to (\mathrm{Cyl}(f'), Z')$. We prove now by induction that it can be homotoped into Z', so let us assume that we have already managed to change $j \circ f$ into a homotopic map h_n which sends the n-skeleton $Z^{(n)}$ into Z'. Consider next the following diagram

$$\begin{array}{ccccc} \bigvee_\alpha S^n_\alpha & \xrightarrow{\vee g_\alpha} & Z^{(n)} & \xrightarrow{j\circ f} & Z' \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle i} \\ \bigvee_\alpha D^{n+1}_\alpha & \xrightarrow[\vee G_\alpha]{} & Z^{(n+1)} & \xrightarrow[h_n]{} & \mathrm{Cyl}(f') \end{array}$$

Here the maps g_α are the attaching maps for the $(n+1)$-dimensional cells and we use abusively the same name for a map and its restriction to a subspace. For each index α we have a relative map $(D^{n+1}, S^n) \to (\mathrm{Cyl}(f'), Z')$.

Now we use the fact that f' is a CW-approximation, it induces isomorphisms on all homotopy groups. So does the homotopy equivalence r and hence the cofibration i is also a weak homotopy equivalence. In particular the relative homotopy group $\pi_{n+1}(\mathrm{Cyl}(f'), Z') = 0$, so that the Compression Lemma 4.7 allows us to change the map of pairs into one to (Z', Z') relative to S^n_α. The homotopies for all cells being constantly equal to g_α on the boundaries, they assemble into a homotopy on $Z^{n+1)}$, from h_n to a map h_{n+1} only defined on $Z^{(n+1)}$ for the moment.

We define h_{n+1} on the entire space Z by extending the homotopy, defined now only on $Z^{(n+1)} \times I$, to $Z \times I$ by the HEP since the inclusion of a skeleton is a cofibration by Theorem 3.9. The map h is then defined on the entire CW-complex Z by setting $h(z) = h_n(z)$ if z belongs to the n-skeleton $Z^{(n)}$. Since h is homotopic to $j \circ f$, which is a weak equivalence, so if h.

We move on finally to the proof of the uniqueness of h. Suppose we have two maps $h_1, h_2 \colon Z \to Z'$ such that $i \circ h_1$ and $i \circ h_2$ are both homotopic to $j \circ f$. There is then a pointed homotopy $H \colon (Z \times I, z_0 \times I \cup Z \times \partial I) \to (\mathrm{Cyl}(f'), Z')$. The target pair is composed of an inclusion which is a weak equivalence. The same argument as above with the help of the Compression Lemma 4.7 allows us to conclude that we can replace H by a homotopy H' entirely contained in Z'. □

We arrive at an important observation about CW-complexes. It seems up to now that we would like to work with homotopy equivalences, but then we have only been able to replace a space up to *weak equivalence* by a nice CW-complex. Since we are interested in understanding homotopy groups of spaces, this is not such a bad trade off, but what if we work with CW-complexes and homotopy equivalences? Henry Whitehead realized that weak equivalences between CW-complexes are actually even better, they are honest homotopy equivalences.

THEOREM 4.3. WHITEHEAD THEOREM. *A weak homotopy equivalence between CW-complexes is a homotopy equivalence.*

PROOF. Let $f\colon X \to Y$ be a weak equivalence between two path-connected CW-complexes (if they are not path-connected, we deal with one component at a time). Then both X and Y, via the identity map, are CW-approximations of Y. The previous Proposition 4.2 tells us that there exists a map $h\colon Y \to X$ such that $f \circ h$ is homotopic to id_Y. Let us look now at the other composition $h \circ f$. When precomposing with f we see that $f \circ h \circ f$ is homotopic to $id_Y \circ f = f$. Using again Proposition 4.2 but this time for the CW-approximations f and f, we conclude by the uniqueness part that $f \circ h$ and id_X are homotopic. Thus f and h are homotopy inverses to each other. □

A direct consequence is that Proposition 4.2 upgrades to a uniqueness up to homotopy of the CW-approximation.

COROLLARY 4.4. *The CW-approximation of a space is unique up to homotopy.*

PROOF. We have seen that there is always a weak equivalence $h\colon Z \to Z'$ between two CW-approximations of the same space X. We deduce from Whitehead's Theorem 4.3 that h is a homotopy equivalence. □

In the spirit of the proof of the Whitehead Theorem 4.3 here is a very useful criterion to recognize homotopy equivalences. Whereas a weak equivalence is detected by homotopy classes of maps out of spheres, we need to allow for more spaces in the source or in the target to detect homotopy equivalences.

PROPOSITION 4.5. *A map $a\colon A \to A'$ is a homotopy equivalence if and only if it induces a bijection $a^*\colon [A', X] \cong [A, X]$ for any space X.*

PROOF. If a is a homotopy equivalence, then it admits a homotopy inverse a'. By functoriality of $[-, X]$, we see that both composition $a^* \circ (a')^*$ and $(a')^* \circ a^*$ are equal to the identity.

To prove the converse, let us assume that a^* is a bijection on homotopy classes of maps for any space X. We will use $X = A$ first so that the bijection $a^*\colon [A', A] \cong [A, A]$ yields a map $a'\colon A' \to A$ such that $a^*[a'] = [id_A]$. This means that $a' \circ a \simeq id_A$. To conclude we have to show that the other composition $a \circ a'$ is homotopic to $id_{A'}$.

Let us choose $X = A'$ this time and compute $a^*[a \circ a'] = [a \circ a' \circ a] = [a] = a^*[id'_A]$. Therefore $[a \circ a'] = [id_A]$ and we are done. □

5. Properties of cofibrations

Let us come back to cofibrations and mention a few important properties. To show all of them would represent too much work compared to the objectives of this course, so there is one powerful feature we will leave as a black box. Interested students can find complete proofs in Strøm's original work [**9**], tom Dieck's book [**12**] or May's [**4**]. In Strøm's following article, [**10**] he actually proves that the category of all spaces forms a so called model category.

We start with a strictification result. It says that one can render a homotopy commutative triangle strictly commutative.

LEMMA 5.1. STRICTIFICATION. *Let $i\colon A \hookrightarrow B$ be a cofibration and $f\colon A \to X$ be a map. Assume that $g\colon B \to X$ is a map such that $g \circ i$ and f are homotopic. Then there exists a map $\tilde{g}\colon B \to X$ such that $\tilde{g} \simeq g$ and $g \circ i = f$.*

PROOF. Since $g \circ i$ and f are homotopic, there exists a homotopy $F\colon A \times I \to X$ from $g \circ i$ to f. Consider now the following extension problem:

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & & \\ \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i_0} & \searrow{\scriptstyle g} & \\ A \times I & \xrightarrow{i \times I} & B \times I & \overset{G}{\dashrightarrow} & X \\ & \searrow_{F} & & \nearrow & \end{array}$$

Since i is a cofibration, the extension G exists. It is a homotopy that starts at g and ends at some map $\tilde{g}$ such that $\tilde{g} \circ i = G(i(-), 1)$, but this is $F(-, 1) = f$. □

Another useful property is that the product $i \times C$ of a cofibration with a given space C is again cofibration.

LEMMA 5.2. *Let $i\colon A \hookrightarrow B$ be a cofibration. Then $i \times C$ is again a cofibration for any space C.*

PROOF. A homotopy extension problem for $i \times C$ into a space X corresponds by adjunction to a homotopy extension problem for i into the mapping space $\mathrm{map}(C, X)$. □

We continue next with this property of cofibrations we will not prove. It is related to the so called "pushout-product" map. Let $f\colon A \to B$ and $g\colon X \to Y$ be two maps and consider the commutative square obtained as follows:

$$\begin{array}{ccc} A \times X & \xrightarrow{A \times g} & A \times Y \\ {\scriptstyle f \times X}\downarrow & & \downarrow{\scriptstyle f \times Y} \\ B \times X & \xrightarrow{B \times f} & B \times Y \end{array}$$

We call P the pushout of the diagram consisting of the left and top arrows.

DEFINITION 5.3. Let $f\colon A \to B$ and $g\colon X \to Y$ be two maps. The pushout $P = \mathrm{colim}(B \times X \xleftarrow{f \times X} A \times X \xrightarrow{A \times g} A \times Y)$ admits a map $f \Box g\colon P \to B \times Y$ called the *pushout-product map*.

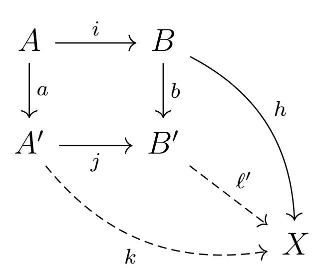
When f and g are cofibrations, then the pushout product map $f \Box g\colon P \to B \times Y$ is a cofibration as well, and if either f or g is a homotopy equivalence, so is $f \Box g$. Even in the very specific case where $g = \partial\colon \{0;1\} \hookrightarrow I$ is the inclusion of the boundary of an interval, the proof is not obvious. Here the pushout is $(A \times I) \cup (B \times \{0;1\}$ and a map out of it to a space X is the data of *two* maps $B \to X$ together with a homotopy $A \times I \to X$ between their restrictions to A. One reason one could believe this, is that for two maps f and g that are sub-CW-complex inclusions, the pushout-product map is another sub-CW-compex inclusion, hence a cofibration. Anyhow, we will use the following fact without proof and record it here for future reference.

Proposition 5.4. *Let $f\colon A \to B$ be a cofibration. Then $(A\times I)\cup(B\times\{0;1\} \hookrightarrow B \times I$ is a cofibration, which is a homotopy equivalence when f is so.* □

We have seen that the pushout of a cofibration along any map is again a cofibration, but it is not true in general that the pushout of a homotopy equivalence along an arbitrary map is again a homotopy equivalence. It is true however when we perform the pushout along a cofibration.

Proposition 5.5. Left properness. *Let $f\colon A \to B$ be a cofibration and $a\colon A \to A'$ be a homotopy equivalence. Then the pushout of a along i is a homotopy equivalence $b\colon B \to B'$.*

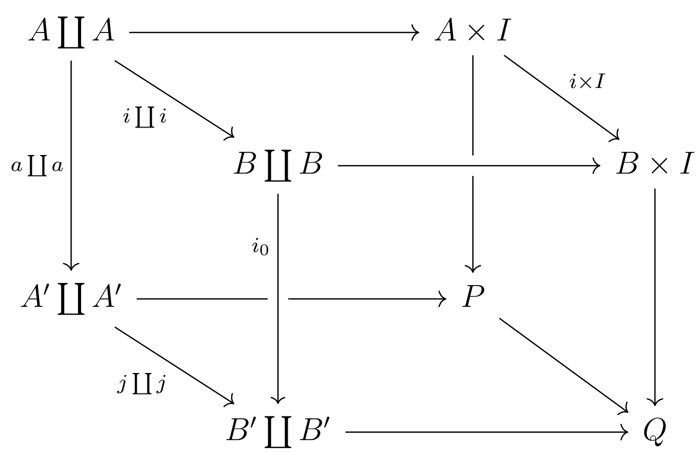
Proof. We check that b is a weak equivalence by verifying that it induces a bijection on homotopy classes of maps into any space X. Diagrammatically we are looking at a diagram of the following form, where the square is a pushout square and $h\colon B \to X$ is any map:



We are looking for the dashed arrow out of B', and for that we will first find the other dashed map k out of A' so as to be able to use the universal property of the pushout. We do not only need to find this map ℓ, we have to show it is unique up to homotopy.

We start with the surjectivity of $b^*\colon [B', X] \to [B, X]$. Since a is a homotopy equivalence, we know that a^* is a bijection, so there exists a map $k\colon A' \to X$ such that $a^*[k] = [h \circ i]$. By the Strictification Lemma 5.1 we can modify h up to homotopy for a map h' such that $k \circ a = h' \circ i$. Now that the diagram commutes strictly we obtain from the universal property of the pushout a unique map $\ell\colon B' \to X$ making the whole diagram commute. In particular $b^*[\ell] = [\ell \circ b] = [h'] = [h]$ and we are done with surjectivity.

We move on to prove injectivity. We first deal with the case when a is not only a homotopy equivalence, but also a cofibration. Assume thus we are given two maps $\ell, \ell'\colon B' \to X$ such that the composition $\ell \circ b$ and $\ell' \circ b$ are homotopic. There exists thus a homotopy $H\colon B \times I \to X$ from $\ell \circ b$ to $\ell' \circ b$. We construct the pushout-product maps $a \Box \partial$ and $b \Box \partial$, let us visualize the spaces P and Q on a cubical diagram:



The left hand side face is a pushout square because it is coproduct of two copies of our original pushout square defining B'. So are the front and back face as we define the spaces P and Q in order to have this. Composition of pushouts yields another pushout so that the square

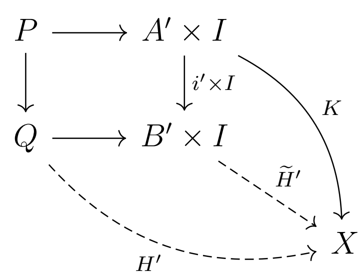
$$\begin{array}{ccc} A \coprod A & \longrightarrow & B \times I \\ {\scriptstyle a \coprod a}\downarrow & & \downarrow \\ A' \coprod A' & \longrightarrow & Q \end{array}$$

is a pushout square as well. Finally, from this last pushout and the one defining P we deduce that the right hand side face is again a pushout square. Let us compose this last one with the pushout product maps:

$$\begin{array}{ccccc} A \times I & \longrightarrow & P & \xrightarrow{a\Box\partial} & A' \times I \\ \downarrow{\scriptstyle i\times I} & & \downarrow & & \downarrow \\ B \times I & \longrightarrow & Q & \xrightarrow{b\Box\partial} & B' \times I \end{array}$$

We know that taking the product with I converts pushout squares into pushout squares, so the large rectangle is a pushout. The same argument as above shows now that so is the right hand side square. We started with a homotopy H on $B \times I$ between two maps $\ell \circ b$ and $\ell' \circ b$. By construction of Q we have thus an induced map $H' \colon Q \to X$. Now $H \circ (i \times I)$ defines a homotopy on A and since $a \times I$ is a homotopy equivalence, we know that $[A' \times I, X] \cong [A \times I, X]$. This homotopy corresponds thus to a homotopy $K \colon A' \times I \to X$ such that $H \circ (i \times I) \simeq K \circ (a \times I)$.

We have seen in Lemma 5.2 that $a \times I$ is a cofibration, so we can apply the Strictification Lemma 5.1 so as to change K up to homotopy and get a strict equality $H \circ (i \times I) = K \circ (a \times I)$. We have now a commutative diagram



To verify that it is indeed commutative we use the universal property of the pushout P and verify the restrictions to $A' \coprod A'$ and $A \times I$ agree. On the coproduct we find $\ell \circ i' \coprod \ell' \circ i'$ and on the cylinder we get $H \circ (i \times I)$. Our previous work showed that the square above is a pushout square, so the homotopy $\widetilde{H}$ is uniquely determined by K and H'. This is precisely a homotopy from ℓ to ℓ' so we are done.

Done? Not quite, we have only solved the problem when the homotopy equivalence a was a cofibration. In general we can reduce to the previous situation by factoring a as $A \hookrightarrow \mathrm{Cyl}(a) \xrightarrow{p} A'$. As we know how to solve the problem for the inclusion into the cylinder, we are left with the case of a strong homotopy retraction. We have a cofibration $A' \hookrightarrow A''$ (the inclusion into the bottom of the cylinder) followed by the collapse map $p\colon A'' \to A'$. Taking the pushout along a cofibration $A' \hookrightarrow B'$ yields first a homotopy equivalence $B' \to B''$ by the previous case, and then a retraction $B'' \to B'$ since the pushout of the identity is the identity. Thus also $B'' \to B'$ is a homotopy equivalence by the recognition principle Proposition 4.5. Indeed we have a factorization of the identity $[B', X] \cong [B'', X] \to [B', X]$ by a bijection followed by ... another bijection. This time we are done! □

REMARK 5.6. This last part about arbitrary homotopy equivalences is not quite correct. I am not sure how to fix it at the moment.

6. Homotopy pushouts

Let I be the category with three objects and two non-identity morphisms $2 \leftarrow 0 \to 1$ so that functors $F\colon I \to Top$ are pushout diagrams. We have an adjunction $\mathrm{colim}_I\colon Top^I \leftrightarrows Top\colon c$ where c is the constant diagram functor sending a space X to the pushout diagram $X = X = X$. Then, for any pushout diagram F, maps of spaces

$$\mathrm{colim}_I F \to X$$

correspond to natural transformations of diagrams $h\colon F \to cX$. This works well categorically, but not so much when we add homotopy to the picture.

EXAMPLE 6.1. Consider the following natural transformation of pushout diagrams $\eta\colon F \to F'$ as described by the following (strictly) commutative diagram

$$\begin{array}{ccccc} D^n & \xleftarrow{i} & S^{n-1} & \xrightarrow{i} & S^{n-1} \\ \downarrow{\simeq} & & \downarrow{id} & & \downarrow{\simeq} \\ * & \longleftarrow & S^{n-1} & \longrightarrow & * \end{array}$$

All vertical maps are homotopy equivalences, so the natural transformation η deserves to be call a (pointwise) equivalence of pushout diagrams. However, when taking pushouts, η induces a map $S^n \to *$ which is not anymore a homotopy equivalence.

This problem motivates us to adapt the construction of pushouts so as to get a coherent homotopy type, one which is homotopy meaningful, i.e. invariant for equivalent diagrams.

DEFINITION 6.2. Let $F = (C \overset{g}{\leftarrow} A \overset{f}{\rightarrow} B)$ be a pushout diagram. Turn f and g into a cofibration followed by a homotopy equivalence $A \hookrightarrow B' \rightarrow B$ and $A \hookrightarrow C' \rightarrow C$ respectively, The *homotopy pushout* of F is the colimit of the diagram $F' = (C' \overset{j}{\leftarrow} A \overset{i}{\rightarrow} B')$. We write $\mathrm{hocolim}_I F$ for the colimit $\mathrm{colim}_I F'$.

REMARK 6.3. By construction we have a natural transformation $F' \rightarrow F$, therefore an induced map $\eta\colon \mathrm{hocolim}_I F \rightarrow \mathrm{colim}_I F$. There is a whole theory of so called *total left derived functors* in homotopy theory that allows one to prove that not is the homotopy colimit a homotopy invariant functors, i.e. its value does not depend on the choices we made to replace the maps f and g by cofibrations and it sends homotopy equivalent diagrams to homotopy equivalent spaces, but it is in fact the "best" such functor, by which we mean that any natural transformation from a homotopy invariant functor to the colimit actually factors through η.

We continue with a few examples and models that are good to have in mind when thinking about homotopy pushouts.

EXAMPLE 6.4. The *double mapping cylinder* of the pushout diagram is the standard model for a homotopy pushout. Given a pushout diagram $F = (C \overset{g}{\leftarrow} A \overset{f}{\rightarrow} B)$, we use the mapping cylinder to turn both maps into cofibrations so as to obtain $F' = (\mathrm{Cyl}(g) \overset{j}{\leftarrow} A \overset{i}{\rightarrow} \mathrm{Cyl}(f))$.

For example, when $C = *$ we get the pushout of $CA \overset{j}{\leftarrow} A \overset{i}{\rightarrow} \mathrm{Cyl}(f))$. This is a version of the mapping cone of f, where the cone on A is a cylinder on A on which we glue a cone. Up to reparametrization, this is $C(f)$, also called the *homotopy cofiber* of f.

When also $A = *$, we get the pushout of $CA \overset{j}{\leftarrow} A \overset{i}{\rightarrow} CA$, which is the suspension ΣA. Depending on the category we are working in, we get the unreduced suspension, or the reduced suspension (for pointed spaces).

Our main result in this section is that homotopy pushouts are homotopy invariant. For this we need two lemmas.

LEMMA 6.5. *Let us consider a diagram where the right hand side square is a pushout:*

$$\begin{array}{ccccc} A_3 & \longleftarrow & A_0 & \longrightarrow & A_2 \\ \downarrow & & \downarrow & & \downarrow \\ A_3' & \longleftarrow & A_1 & \longrightarrow & A_{12} \end{array}$$

and complete it to a cube by constructing the pushouts A_{23} of the top row, and A_{23}' of the bottom row. Then the front face

$$\begin{array}{ccc} A_3 & \longrightarrow & A_{23} \\ \downarrow & & \downarrow \\ A_3' & \longrightarrow & A_{23}' \end{array}$$

is also a pushout square.

PROOF. We obtain a unique map $A_{23} \to A_{23}'$ making the whole diagram commute, so the situation makes sense. Composing the original pushout square with the bottom pushout square yields a pushout square

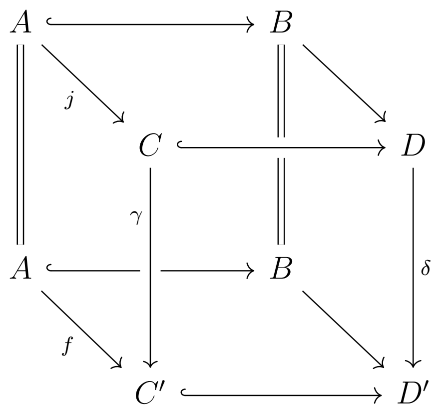
$$\begin{array}{ccc} A_0 & \longrightarrow & A_3' \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & A_{23}' \end{array}$$

Since the top face is also a pushout square, so is the front face. $\square$

The next lemma is already closely related to our main theorem.

LEMMA 6.6. *Let D' be the pushout of a diagram $C' \overset{f}{\leftarrow} A \hookrightarrow B$ and factor f as $A \hookrightarrow C \xrightarrow{\simeq} C'$. The homotopy equivalence $\gamma\colon C \to C'$ induces then a homotopy equivalence of pushouts $D = \operatorname{colim}(C \leftarrow A \to B) \to D'$.*

PROOF. We apply Lemma 6.5 to the following cube



and deduce that the front face is a pushout square. By left properness, Proposition 5.5, we conclude that δ is a homotopy equivalence. □

We are ready to prove the homotopy invariance of the homotopy pushout construction: it does not depend on the choice of the replacement of our maps by a cofibration. In fact, thanks to the previous lemma, we only need to change one single map by a cofibration.

THEOREM 6.7. HOMOTOPY INVARIANCE OF PUSHOUTS. *Consider a natural transformation of pushout diagrams which is pointwise a homotopy equivalence:*

$$\begin{array}{ccccc} C & \xleftarrow{f} & A & \xhookrightarrow{i} & B \\ \simeq\big\downarrow\gamma & & \simeq\big\downarrow\alpha & & \simeq\big\downarrow\beta \\ C' & \xleftarrow{f'} & A' & \xhookrightarrow{i'} & B' \end{array}$$

Assume that i and i' are cofibrations. Then $\delta\colon D \to D'$ is a homotopy equivalence.

PROOF. Factoring simultaneously f and f' into a cofibration followed by a homotopy equivalence, we can use Lemma 6.6 so as to reduce the proof to the case where all horizontal maps, including f and f' are cofibrations. Then we factor the vertical natural transformation by inserting the pushout P of i and α:

$$\begin{array}{ccccc} C & \xleftarrow{\;f\;} & A & \xhookrightarrow{\;i\;} & B \\ {\simeq}\downarrow{\gamma} & & {\simeq}\downarrow{\alpha} & & {\simeq}\downarrow{\pi} \\ C' & \xleftarrow{\;f'\;} & A' & \hookrightarrow & P \\ \| & & \| & & {\simeq}\downarrow \\ C' & \xleftarrow{\;f'\;} & A' & \xhookrightarrow{\;i'\;} & B' \end{array} \qquad \beta : B \to B'$$

We observe that the pushout of α along the cofibration i is again a homotopy equivalence we called π in the diagram above. From the universal property of the pushout P we have a unique map $P \to B'$ which is also a homotopy equivalence because β and π are so.

We apply now Lemma 6.5 and construct the pushout Q of the middle row and conclude that the front face

$$\begin{array}{ccc} C & \hookrightarrow & D \\ \downarrow & & \downarrow \\ C' & \hookrightarrow & Q \end{array}$$

is also a pushout square. By left properness the comparison map $D \to Q$ is a homotopy equivalence. Moreover Lemma 6.6 implies that so is the map $Q \to D'$. The composite $D \to D'$ is thus a homotopy equivalence as well and we are done. □

REMARK 6.8. It does not matter whether we compare two pushout diagrams having a cofibration on the same side, like we stated it in the theorem, or on different sides. The proof goes through in the same way. Let us also say that formally the homotopy invariance of the homotopy pushout construction is equivalent to left properness.

7. Playing with homotopy pushouts

To illustrate the way one can use homotopy pushouts, we will consider in this section "pushout diagrams of pushout diagrams" and provide two models to compute them. Sometimes one is interesting and the other one is easy to compute! Let us start with a property about strict colimits. Later we will prove the analogous version for homotopy colimits.

Proposition 7.1. Fubini for colimits. *Let I and J be small categories and $F\colon I \times J \to Top$ be a functor. Then*

$$\mathrm{colim}_{j\in J}[\mathrm{colim}_I F(-,j)] \approx \mathrm{colim}_{i\in I}[\mathrm{colim}_J F(i,-)]$$

Proof. Both colimits of colimits are models for the colimit of F, taken on the product category $I \times J$, they verify the universal property. □

Example 7.2. Let $A \subset X$ and $B \subset Y$ be inclusions of pointed subspaces. Then $(X \vee Y)/(A \vee B) \approx X/A \vee Y/B$. To see that we draw a three-by-three diagram corresponding to a pushout of puhsout diagrams and indicate by squiggly arrows the computation of the corresponding horizontal pushouts:

$$\begin{array}{ccccccc}
* & = & * & = & * & \rightsquigarrow & * \\
\uparrow & & \| & & \uparrow & & \uparrow \\
B & \longleftarrow & * & \longrightarrow & A & \rightsquigarrow & A \vee B \\
\downarrow & & \| & & \downarrow & & \downarrow \\
Y & \longleftarrow & * & \longrightarrow & X & \rightsquigarrow & X \vee Y
\end{array}$$

Hence the vertical pushout of these horizontal pushouts is the quotient $(X \vee Y)/(A \vee B)$. If we start instead with vertical puhsouts we find the diagram $Y/B \leftarrow * \rightarrow X/A$ whose pushout is $X/A \vee Y/B$.

We wish to obtain an analogous result for homotopy pushouts and focus therefore on the case $I = J = 2 \leftarrow 0 \rightarrow 1$. A diagram indexed by $I \times I$ is therefore of the form:

$$\begin{array}{ccccc}
A_{12} & \longleftarrow & A_{10} & \longrightarrow & A_{11} \\
\uparrow & & \uparrow & & \uparrow \\
A_{02} & \longleftarrow & A_{00} & \longrightarrow & A_{01} \\
\downarrow & & \downarrow & & \downarrow \\
A_{22} & \longleftarrow & A_{20} & \longrightarrow & A_{21}
\end{array}$$

In order to compute homotopy pushouts we need to turn all maps into cofibrations (so we can keep A_{00} unchanged), but as we will also iterate, let us make the corner spaces even fatter so as to contain nicely the pushout of each square. More concretely we turn first every map $\alpha\colon A_{00} \to A_{ij}$, with i or j equal to zero, into a cofibration followed by a homotopy equivalence:

$$A_{00} \hookrightarrow \mathrm{Cyl}(\alpha) = A'_{ij} \xrightarrow{\simeq} A_{ij}$$

We obtain an equivalent diagram with a natural transformation to the original diagram made of homotopy equivalences pointwise:

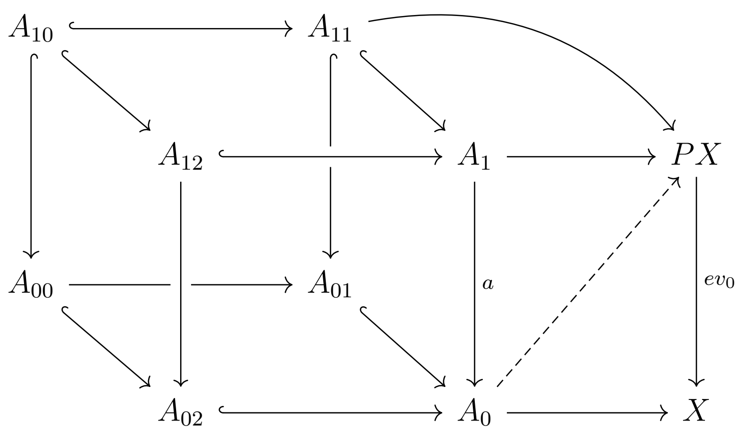
$$\begin{array}{ccccc}
A_{12} & \longleftarrow & A'_{10} & \longrightarrow & A_{11} \\
\uparrow & & \uparrow & & \uparrow \\
A'_{02} & \hookleftarrow & A_{00} & \hookrightarrow & A'_{01} \\
\downarrow & & \downarrow & & \downarrow \\
A_{22} & \longleftarrow & A'_{20} & \longrightarrow & A_{21}
\end{array}$$

As mentioned above this would be good enough to compute either horizontal pushouts or vertical ones, but we let P_{ij} be the pushout of the square with terminal corner A_{ij}, obtain a map $P_{ij} \to A_{ij}$ by the universal property, turn it into a cofibration followed by a homotopy equivalence $P_{ij} \hookrightarrow A'_{ij} \xrightarrow{\simeq} A_{ij}$. We finally obtain another pointwise equivalent diagram replacing A_{ij} by A'_{ij} for $ij \neq 0$.

Let us look more closely at horizontal pushouts (the case of vertical pushouts is completely analogous). We call A_i the pushout of the diagram $A_{i2} \hookleftarrow A_{i0} \hookrightarrow A_{i1}$.

Lemma 7.3. *In the above situation taking horizontal pushouts induces cofibrations* $A_0 \hookrightarrow A_i$ *for* $i = 1, 2$.

Proof. Instead of checking the HEP by looking at cylinders on all spaces that are involved, we look at the equivalent problem by adjunction and will construct an extension to $\mathrm{map}(I, X) = PX$, the space of paths in X. Let us look for example at the map $A_0 \to A_1$. To show that it is a cofibration we consider a homotopy extension problem of the following form, where we also draw the horizontal pushout constructions:



By precomposing the HEP to A_{01} we find an extension $A_{01} \to PX$ since the vertical maps $A_{11} \hookrightarrow A_{01}$ is a cofibration. By commutativity of the whole diagram this homotopy, together with the one we already have $A_{12} \to PX$ yield a map $P_{02} \to PX$ out of the pushout of the left hand side face.

Now we use the assumption on our diagram that the map $P_{02} \hookrightarrow A_{02}$ is a cofibration as well. This gives us an extension $A_{02} \to PX$ which is compatible with the previous homotopies. Since A_0 is the pushout of the bottom face of the cube, we finally find the desired (dashed) extension $A_0 \to PX$. $\square$

THEOREM 7.4. FUBINI FOR HOMOTOPY PUSHOUTS. *Let $A_{\bullet\bullet}\colon I \times I \to Top$ be a functor. Then*

$$\mathrm{hopo}_{j\in I}[\mathrm{hopo}_I A_{\bullet j}] \simeq \mathrm{hopo}_{i\in I}[\mathrm{hopo}_I A_{i\bullet}]$$

PROOF. By replacing the diagram as explained above we find ourselves in a position where all homotopy pushouts appearing in the formula we wish to prove are given by strict pushouts. The fact that the second round of homotopy pushout are also strict ones is given by the previous Lemma 7.3. We can then apply Proposition 7.1 to conclude. $\square$

We end this section with one simple, but useful trick.

EXAMPLE 7.5. Consider a pair of composable maps $A \xrightarrow{f} B \xrightarrow{g} C$. We do not require anything about these maps. We claim that there is always a homotopy cofiber sequence

$$C(f) \xrightarrow{\alpha} C(g \circ f) \to C(g)$$

where the map α is induced by g. In other words we claim that the homotopy cofiber $C(g)$ is homotopy equivalent to the homotopy cofiber of α. For this, no surprise, we use a 3×3-diagram:

$$\begin{array}{ccccccc}
* & = & * & = & * & \rightsquigarrow & * \\
\| & & \uparrow & & \uparrow & & \uparrow \\
* & \longleftarrow & A & \xrightarrow{f} & B & \rightsquigarrow & C(f) \\
\| & & \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \alpha} \\
* & \longleftarrow & A & \xrightarrow{g \circ f} & C & \rightsquigarrow & C(g \circ f)
\end{array}$$

The squiggly arrow indicate now *homotopy pushouts* and we can compare the homotopy cofiber of α by computing the same homotopy type with our Fubini Theorem 7.4. We get, up to homotopy a diagram $* \leftarrow * \to C(g)$ whose homotopy pushout is $C(g)$.

8. Fibrations

Now that we have studied cofibrations and played with homotopy pushouts, we come back to the study of the category of spaces and dualize what we have done for cofibrations. Whereas cofibrations can be thought of as nice subspace inclusions, we think of fibrations as nice projections. The proofs in this sections will be more like sketches of proofs since they are formally dual to the ones we have written down for cofibrations. We start with the dual of the HEP.

Definition 8.1. A map $p\colon E \to B$ has the *homotopy lifting property* (HLP) for the space X if for any map $f\colon E \to E$ and homotopy $H\colon X \times I \to B$ starting at $p \circ f$ there is a homotopy $F\colon X \times I \to E$ lifting H and starting at f.

Here is a diagram that explains better than words the lefting problem we wish to solve:

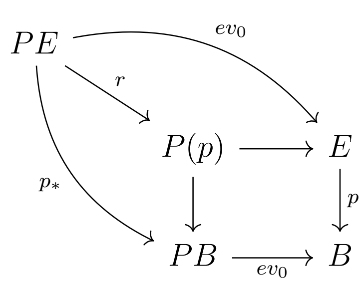
$$\begin{array}{ccc}
X & \xrightarrow{f} & E \\
\downarrow{\scriptstyle i_0} & \overset{F}{\nearrow} & \downarrow{\scriptstyle p} \\
X \times I & \xrightarrow[H]{} & B
\end{array}$$

Definition 8.2. A map p is a *(Hurewicz) fibration* if it has the HLP for all spaces X.

Example 8.3. Let X be a non-empty space and $p_2\colon X \times B \to B$ be the projection onto the second factor. This is a fibration since a lift of a homotopy $X \times I \to B$ starting at $p_2 \circ f$ can always be obtained by defining the lift component by component. We just set $F(x,t) = (p_1(f(x)), H(x,t))$.

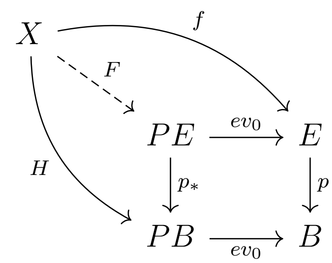
Remark 8.4. The three classes of maps, homotopy equivalences, (Hurewicz) fibrations, and (Hurewicz) cofibrations, equip the category of topological spaces with a so-called Quillen mode structure. It is called the Strøm or the Hurewicz model structure (I prefer the first since Strøm was the one to prove it is a model structure in 1972, [**10**]). There is another model structure for weak equivalences, Serre fibrations, which enjoy the HLP for all cubes I^n, but are not required to have it for all spaces, and cofibrations are retracts of relative cell complexes.

Just like for cofibrations one does not need to establish the HLP for all spaces, one universal space is enough. We used a cylinder for cofibrations, here we need a path space $P(p) = \{(e, \omega) \in E \times PB \mid p(e) = \omega(0)\}$. In other words $P(p)$ is the pullback in the following diagram and the universal property explains it comes with a map from the path space $PE = \mathrm{map}(I, E)$:



Proposition 8.5. *A map $p\colon E \to B$ is a fibration if and only if the map $r\colon PE \to P(p)$ admits a section $s\colon P(p) \to PE$.*

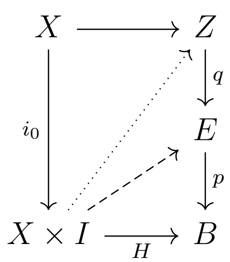
Proof. By adjunction we see homotopies in E as maps into PE, so we can rewrite the HLP as follows:

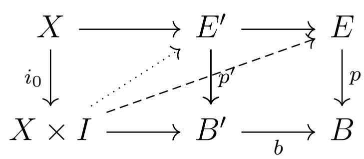


Since $ev_0 \circ H = p \circ f$ we see that the homotopy H starts at $p \circ f$. If p is a fibration, then F must exist when $X = P(p)$, we call $s = F$. Conversely if s exists, then we can choose $F = s \circ h$, where $h\colon X \to P(p)$ is the map given by the universal property of the pullback for f and H. □

Proposition 8.6. *Fibrations are stable under composition and pullback.*

Proof. Let us simply draw the diagrams that explain the arguments behind both properties:



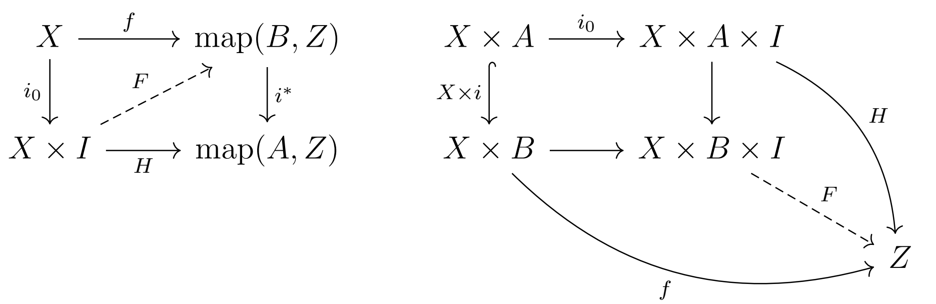


In the first diagram the maps p and q are composable fibrations, in the second one the map p' is the pullback of p along b. In both cases one constructs the desired lift in two steps, first the dashed arrow, then the dotted one. □

One new feature we see in this section is due to our previous study of cofibrations and the adjunction with mapping spaces. As usual, but let us recall this here since it has been some time we have not said this, this only applies to locally conpact and Hausdorff spaces.

Proposition 8.7. *Let $i\colon A \hookrightarrow B$ be a cofibration and Z be any space. Then $i^*\colon \mathrm{map}(B, Z) \to \mathrm{map}(A, Z)$ is a fibration.*

PROOF. By adjunction the lifting problem on the left corresponds to the extension problem on the right:



Since i is a cofibration, so is $X \times i$. Therefore the extension H exists in the right hand side diagram. It corresponds to the adjoint map, abusively written H on the left. □

Important examples of such fibrations are evaluation maps. We have already met them, but now we know they are fibrations.

EXAMPLE 8.8. The inclusion $\partial I \subset I$ and the inclusion $0 \subset I$ are cofibrations. The evaluation maps $ev_0 \colon \mathrm{map}(I, X) \twoheadrightarrow X \approx \mathrm{map}(0, X)$ and

$$\mathrm{ev}_{01} \colon \mathrm{map}(I, X) \twoheadrightarrow X \times X$$

are therefore fibrations. The second one corresponds to the evaluation at 0 and 1, so for any path $\omega \colon I \to X$, we have $\mathrm{ev}_{01}(\omega) = (\omega(0), \omega(1))$.

When (X, x_0) is a pointed space we can pullback ev_{01} along the inclusion $X \approx x_0 \times X \hookrightarrow X \times X$ and obtain by stability under pullbacks, Proposition 8.6, that $ev_1 \colon \mathrm{map}_*(I, X) \to X$, the evaluation at 1 from the pointed path space (of those paths in X starting at the base point) is again a fibration.

Just like the mapping cylinder helped us to replace an arbitrary map by a cofibration, dually path spaces help us to replace any map by a fibration.

PROPOSITION 8.9. *Let $f \colon X \to Y$ be any map. Then there exists a factorization $f \colon X \xrightarrow{\simeq} P(f) \xrightarrow{q} Y$ into a homotopy equivalence followed by a fibration.*

Proof. Let us construct the pullback P in the following diagram

$$\begin{array}{ccc} P & \longrightarrow & PY \\ {\scriptstyle q'}\big\downarrow & & \big\downarrow{\scriptstyle ev_{01}} \\ X \times Y & \xrightarrow[f\times Y]{} & Y \times Y \end{array}$$

By construction of a pullback, the points of P are triples $(x, y, \omega) \in X \times Y \times PY$ such that $ev_{01}(\omega) = (f(x), y)$. This means that $\omega(0) = f(x)$ and $\omega(1) = y$. The extra information in the second component is thus superfluous, since it is determined by ω, so P is actually homeomorphic to $P(f)$, the subset of $X \times PY$ of pairs (x, ω) such that $\omega(0) = f(x)$.

The pullback of a fibration is a fibration by Proposition 8.6, so $q' \colon P(f) \to X \times Y$ is a fibration. Explicitely $q'(x, \omega) = (x, \omega(1))$. To see that we have first to use the above homeomorphism, i.e. identify (x, ω) with $(x, \omega(1), \omega) \in P$ and project to the first two components. The projection $X \times Y \to Y$ is also a fibration, see Example 8.3, so the composition $q \colon P(f) \xrightarrow{q'} X \times Y \to Y$ is again a fibration by Proposition 8.6.

To find the desired factorization we finally choose $X \to P(f)$ to be given by $x \mapsto (x, c_{f(x)})$. This a homotopy equivalence juste like the inclusion $Y \to \mathrm{Cyl}(f)$ is one dually. For an explicit proof, you can have a look at [**12**, 5.7]. The homotopy inverse is given by the pullback of the evaluation fibration $ev_0 \colon PY \twoheadrightarrow Y$ along f. □

We will not prove all the facts we have established for cofibrations: right properness, homotopy pullbacks, homotopy fibers, Fubini Theorem for homotopy pullbacks, etc, but you can easily imagine what this is all about.

9. Properties and examples of fibrations

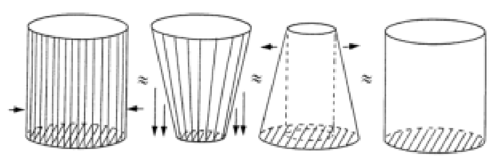
Our aim in this section is to relate the long exact sequence in homotopy of a pair with the homotopy groups of base space and total space of a fibration. In order to do so we have to understand better the preimages of points by a fibration. We follow Bredon's treatment here, see [**1**, Section VII.6]. We start with a lemma about a kind of relative lifting property.

LEMMA 9.1. *Let p be a fibration. Then any solid arrow commutative diagram*

$$\begin{array}{ccc} D^n \times 0 \cup S^{n-1} \times I & \longrightarrow & E \\ \big\downarrow & \nearrow & \big\downarrow p \\ D^n \times I & \longrightarrow & B \end{array}$$

admits a dashed lift.

PROOF. The pair $(D^n \times I, D^n \times 0 \cup S^{n-1} \times I)$ is homeomorphic, as a pair, to $(D^n \times I, D^n \times 0)$. Explicit formulas would not be enlightning, so let us imagine the case $n = 2$ where we have a cylinder on a circle together with the bottom lid as subspace of the solid cylinder. The homeomorphism is done in three steps as drawn by Bredon in his book [**1**, Figure VII-6]:



We first shrink the bottom of the cylinder to say half of its diameter, obtaining a truncated cone. Next we keep the small solid cylinder of radius $1/2$ fixed and project the remaining part down so that the whole vertical boundary corresponding to $S^1 \times I$ is now lying on $D^2 \times 0$, as an annulus of radius $1/2$. Finally we inflate this reversed truncated cone to get back to $D^2 \times I$. The original subspace corresponds precisely to $D^2 \times 0$. $\square$

As a consequence we can use an induction on cells to obtain a more general version of this relative HLP.

PROPOSITION 9.2. *Let p be a fibration and $X \subset Y$ be a sub-CW-complex. Then any solid arrow commutative diagram*

$$\begin{array}{ccc} Y \times 0 \cup X \times I & \longrightarrow & E \\ \big\downarrow & \nearrow & \big\downarrow p \\ Y \times I & \longrightarrow & B \end{array}$$

admits a dashed lift.

The reason we are interested in this is that we obtain more general lifting properties than just for homotopies. Let us consider a strong deformation retract of CW-complexes. This means that we have a subcomplex $i\colon X \hookrightarrow Y$ together with a retract $r\colon Y \to X$ such that $r \circ i = id_X$ and $i \circ r \simeq id_Y$ relative to X. So there is a homotopy $F\colon Y \times I \to Y$ starting at the identity id_Y and ending at $i \circ r$ which on $X \times I$ is $F(x,t) = x$.

THEOREM 9.3. *Let p be a fibration and $X \subset Y$ be a strong deformation retract of CW-complexes. Then any solid arrow commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ \big\downarrow & \nearrow & \big\downarrow p \\ Y & \xrightarrow[g]{} & B \end{array}$$

admits a dashed lift.

PROOF. We insert intermediate steps in the diagram above:

$$\begin{array}{ccccccc} X \times 0 & \xrightarrow{i_0} & X \times I \cup Y \times 1 & \xrightarrow{p_1 \times r} & X & \xrightarrow{f} & E \\ {\scriptstyle i\times 0}\big\downarrow & & \big\downarrow & \overset{H}{\nearrow} & \big\downarrow & & \big\downarrow p \\ Y \times 0 & \xrightarrow[i_0]{} & Y \times I & \xrightarrow[F]{} & Y & \xrightarrow[g]{} & B \end{array}$$

The solid arrow diagram commutes since F is a relative homotopy, the top composition is f and the bottom composition is g because we chose F to start at id_Y. Now, the dashed arrow H exists by Proposition 9.2 and we observe that $H \circ i_0$ solves the lifting problem. □

This property is close to the model categorical property that characterizes fibrations: they enjoy a lifting property with respect to all cofibrations that are also equivalences. For us it will be particularly helpful to compare relative homotopy groups.

THEOREM 9.4. *Let p be a fibration and $B_0 \subset B$ be a subspace containing a chosen base point b_0. Let $E_0 = p^{-1}(B_0) \subset E$ and fix a base point $e_0 \in p^{-1}(b_0)$. Then p induces an isomorphism $p_*\colon \pi_n(E, E_0) \to \pi_n(B, B_0)$.*

PROOF. We prove that p_* is surjective and injective. For surjectivity, let us consider a map $b\colon (D^n, S^{n-1}) \to (B, B_0)$ representing a class $\beta \in \pi_n(B, B_0)$. The dashed lift exists in the following diagram

$$\begin{array}{ccc} * & \xrightarrow{e_0} & E \\ \big\downarrow & \overset{f}{\nearrow} & \big\downarrow p \\ D^n & \xrightarrow[b]{} & B \end{array}$$

since the inclusion of a point in a disc is a strong deformation retract. We know that $b(S^{n-1})$ is contained in B_0, so $f(S^{n-1})$ must lie in the preimage under p, i.e. E_0. The base point has been taken care of, so we found a preimage $[f]$ of β.

We move now to the injectivity. Let f, f' be two maps of pairs such that $p \circ f \simeq p \circ f'$ via a homotopy $F\colon D^n \times I \to B$ restricting on $S^{n-1} \times I$ to a homotopy into B_0. The data we have here, namely f and f', the base point e_0 and the homotopy F correspond to the solid arrow diagram below:

$$\begin{array}{ccc} (D^n \times \partial I) \cup (* \times I) & \xrightarrow{f \coprod f' \cup c_{e_0}} & E \\ \big\downarrow & \overset{H}{\nearrow} & \big\downarrow p \\ D^n \times I & \xrightarrow[F]{} & B \end{array}$$

The left hand side inclusion is a strong deformation retract, so that the dashed lift exists by Theorem 9.3. This is a homotopy of pairs. □

DEFINITION 9.5. Let $p\colon E \to B$ be a fibration and $b_0 \in B$ a (base) point. Then $F_{b_0} = p^{-1}(b_0)$ is the *fiber over* b_0.

COROLLARY 9.6. *Let $p\colon E \to B$ be a fibration and $F = F_{b_0} = p^{-1}(b_0)$ be the fiber over a point $b_0 \in B$. Then $\pi_n(E, F) \cong \pi_n B$ and in particular the long exact sequence for the pair (E, F) can be rewritten as*

$$\cdots \to \pi_{n+1} F \xrightarrow{i_*} \pi_{n+1} E \xrightarrow{p_*} \pi_{n+1} B \xrightarrow{\partial} \pi_n F \xrightarrow{i_*} \pi_n E \to \dots$$

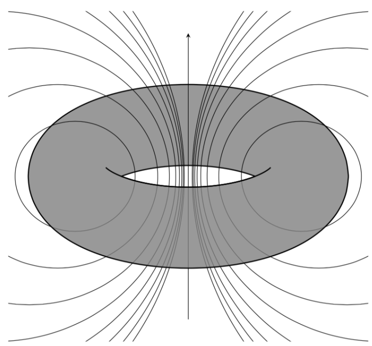
PROOF. For this sequence to make sense we choose compatible base points $f_0 \in F$, $e_0 \in E$. The isomorphism is a direct consequence of Theorem 9.3 for the subspace $B_0 = b_0$. Then we identify the homomorphisms in the long exact sequence of a pair. First the composite

$$\pi_n(E, e_0) \to \pi_n(E, F) \xrightarrow{p_*} \pi_n(B, b_0)$$

is the map induced by p and second the connecting homomorphism can be described as follows. Let $\beta \in \pi_{n+1}B$ be represented by a map $b\colon S^{n+1} \to B$. We precompose it with the map $c\colon D^{n+1} \to S^{n+1}$ collapsing the boundary S^n to a point and lift it to a map $D^{n+1} \to E$ such that S^n is sent to F. This is well-defined up to homotopy by the previous theorem. Then $\partial\beta$ is obtained by restricting precisely to S^n, this yields a map $S^n \to F$ representing $\partial\beta$. □

We conclude this section and this long chapter with an example and a remark.

Example 9.7. The Hopf map $\eta\colon S^3 \to S^2$ is a so-called "fiber bundle", i.e. a map which is locally trivial in the sense that when restricted to each hemisphere in S^2, the Hopf map is homeomorphic to a projection $S^1 \times D^2_\pm \to D^2_\pm$. Indeed we can view S^3 as a union $(S^1 \times D^2_+) \cup_{(S^1 \times S^1)} (D^2_- \times S^1)$



A fiber bundle is a (Serre) fibration and we see here that all fibers are circles, so there is an associated long exact sequence in homotopy:

$$\cdots \to \pi_{n+1}S^1 \xrightarrow{i_*} \pi_{n+1}S^3 \xrightarrow{p_*} \pi_{n+1}S^2 \xrightarrow{\partial} \pi_n S^1 \xrightarrow{i_*} \pi_n S^3 \to \dots$$

We know that all homotopy groups of the circle are trivial, except $\pi_1 S^1 \cong \mathbb{Z}$. Therefore $\pi_n S^3 \cong \pi_n S^2$ for all $n \geq 3$. In particular we see that $\boxed{\pi_3 S^2 \cong \pi_3 S^3 \cong \mathbb{Z}}$ and this infinite cyclic group is generated by the homotopy class of the Hopf map $[\eta]$.

Remark 9.8. At the beginning of the chapter we learned how to associate a long exact sequence to an h-exact sequence and now we just proved that any fibration gives rise to a long exact sequence in homotopy. What is the link between these two types of exact sequences? Maybe it is already clear that both are the same, but let us make the comparison explicit. We compare the fiber of a fibration $p\colon E \to B$

with the homotopy fiber $F(p)$ as follows. Let us assume that the base space B is connected, otherwise we deal with one component at a time, and then fix a base point b_0 (we will see in an exercise that this is a harmless restriction since all fibers have the same homotopy type).

In general, to replace a map $f\colon X \to Y$ by a fibration we take the mapping space $PY = \mathrm{map}(I, Y)$ and pullback the evaluation at 1 along f to get a mapping path space $P(f) = X \times_Y PY$, homotopy equivalent to X. The homotopy fiber $F(f)$ of f is the fiber of the map $P(f) \to Y$ sending a pair (x, ω) to $\omega(0)$. In other words, the homotopy fiber is the subspace of $X \times \mathrm{map}(I, Y)$ of pairs (x, ω) such that $\omega(1) = f(x)$ and $\omega(0) = y_0$. So $F(f)$ is the homotopy pullback of $E \xrightarrow{p} Y \xleftarrow{ev_1} \mathrm{map}_*(I, Y)$. In our situation we have therefore:

$$\begin{array}{ccccccc}
E & \overset{p}{\twoheadrightarrow} & B & \longleftarrow & b_0 & \rightsquigarrow & F \\
\| & & \| & & \downarrow & & \downarrow \simeq \\
E & \underset{p}{\longrightarrow} & B & \overset{ev_1}{\twoheadleftarrow} & \mathrm{map}_*(I, B) & \rightsquigarrow & F(p)
\end{array}$$

the squiggly arrows indicate the pullback operation. Remember that the path space $\mathrm{map}_*(I, B)$ is contractible so all vertical maps are homotopy equivalences. Each row is a pullback diagram in which one map is a fibration, p by assumption and the evaluation ev_1 by Example 8.8. By homotopy invariance of homotopy pullbacks we conclude that $F \simeq P(f)$, the fiber of a fibration is homotopy equivalent to its homotopy fiber.

CHAPTER 5

The Hurewicz homomorphism

This chapter is devoted to our final "big" classical result in homotopy theory. It relates homotopy groups with homology groups. Since we have been dealing with homotopy groups and more generally homotopy classes of maps, we need to come back to the construction of (cellular) homology groups and recast some known results in the light of the homotopy theory we have seen up to now.

1. CW-complexes and homology

Later in this chapter we will use cellular homology and thus CW-complexes, but if we wish to understand general results about all spaces, we need to clarify the relation of homology with respect to weak homotopy equivalences. We know of course that homology groups are homotopy invariants. The next result is also due to Whitehead, it says that homology is a weak homotopy invariant. If we do not indicate the coefficients it will mean that we are considering homology with integral coefficients.

Proposition 1.1. *Let $f\colon X \to Y$ be a weak homotopy equivalence. The induced map $f_* = H_n(f)\colon H_nX \to H_nY$ is then an isomorphism.*

Proof. We turn f into a cofibration $i\colon X \hookrightarrow Y' = \mathrm{Cyl}(f)$, which is also a weak homotopy equivalence. The long exact sequence in homology of the pair (Y', X) shows that it is enough to compute $H_n(Y', X)$ and prove they are all zero. We follow the strategy in Hatcher's book [**3**, Chapter 2], probably well known from the algebraic topology course.

A homology class in represented by a relative cycle $\alpha = \sum k_i\sigma_i$, where the $\sigma_i\colon \Delta^n \to Y'$ are n-simplices whose boundaries assemble to a chain $\partial\alpha$ in X. Construct now a simplicial complex K by assembling all Δ^n's, identifying faces on which the corresponding σ_i's coincide. By the universal property of the quotient, we have then an induced map $\sigma\colon K \to Y'$. The faces of simplices we have used to assemble K do not contribute anything to $\partial\alpha$, but there are other faces appearing in it. We use them to construct a subcomplex $L \subset K$. Thus σ can be considered as a relative map $(K, L) \to (Y', X)$. Moreover, using the obvious singular simplices in K we find a cycle $a \in C_n(K, L)$ such that $\sigma \circ a = \alpha$.

We know that $\pi_n(Y', X) = 0$ and use next the Compression Lemma 4.7, since (K, L) is a finite relative CW-complex, to find a homotopic map of pairs $\tau \simeq \sigma$ factoring through (X, X). Since σ and τ are homotopic, they induce the same map in homology, so $\sigma_*[a] = \tau_*[a] = 0$ because $H_n(X, X) = 0$. □

This justifies that when studying not only homotopy groups, but also homology groups, we might as well concentrate on CW-complexes. Let us end this short introductory section with a result on the skeleta of such a CW-complex.

Lemma 1.2. *Let X be a connected CW-complex, $n \geq 0$, and $X^{(n+k)}$ the $(n+k)$-skeleton for some integer $k \geq 1$. The inclusion $i\colon X^{(n+k)} \hookrightarrow X$ induces isomorphisms $\pi_n X^{(n+k)} \cong \pi_n X$ and $H_n X^{(n+k)} \cong H_n X$.*

Proof. The pair $(X, X^{(n+k)})$ is $(n+k)$-connected as we only attach cells of dimension $> n + k$ to the skeleton to construct X. We see then from the long exact sequence in homotopy that we have an isomorphism for all homotopy groups up to degree $n + k - 1$, in particular on π_n.

For homology we can use cellular homology instead of singular homology. Up to degree $n+k$ the cellular chain complexes of X and $X^{(n+k)}$ are isomorphic since they only depend on the number of cells and their attaching maps: $C_i^{cell}(X) \cong \oplus_{i-cells}\mathbb{Z} \cong C_i^{cell}(X^{(n+k)})$ for $i \leq n+k$. Here as well we obtain an isomorphism for all homology groups up to degree $n + k - 1$. □

Example 1.3. Let $n \geq 2$. We view $S^n = e^0 \cup e^n$ as a CW-complex with two cells, and then $S^n \times S^n$ as a CW-complex with four cells $e^0 \cup e^n \cup e^n \cup e^{2n}$, where each cell corresponds to a product of cells. The $(2n-1)$-skeleton $(S^n \times S^n)^{(2n-1)}$ is thus the wedge $S^n \vee S^n$. We learn from the previous lemma that $\pi_n(S^n \vee S^n) \cong \pi_n(S^n \times S^n) \cong \mathbb{Z} \oplus \mathbb{Z}$ (something we have also seen in an exercise).

More precisely we can identify concrete generators of this free abelian group of rank two, namely the wedge summand inclusion $i_1 \colon S^n \hookrightarrow S^n \vee S^n$ and $i_2 \colon S^n \hookrightarrow S^n \vee S^n$, since their composition into the product yields the inclusion into the two factors of the product, which we know are generators of $\pi_n(S^n \times S^n)$.

The same computation holds in homology, $H_n(S^n \vee S^n) \cong H_n(S^n \times S^n) \cong \mathbb{Z} \oplus \mathbb{Z}$. More generally this argument shows that both the first non-trivial homotopy group and the first non-trivial reduced homology group of an arbitrary wedge $\vee_i S^n$ are isomorphic to a direct sum $\oplus_i \mathbb{Z}$. The wedge summand inclusions provide explicit generators for π_n.

The proof goes by induction for a finite number of spheres. For an infinite number of spheres, the computation can be done directly for H_n, but we need to use the compactness of S^n and notice that any map $S^n \to \vee S^n$ factors through a finite wedge.

Let us finally record a computational simplification which is probably clear at this point, but still useful.

Remark 1.4. Let X be an $(n-1)$-connected space. Then CW-approximation allows us to replace X by a weakly equivalent CW-complex constructed from a point ba attaching cells of dimension $\geq n$. We will thus often assume that such a space can be chosen, up to weak homotopy equivalence, so that

$$X^{(n-1)} = *,\ X^{(n)} = \vee S^n_i,\ X^{(n+1)} = (\vee S^n_i) \cup (\cup_j e^{n+1}_j),\ \cdots$$

In particular, if $f_j \colon S^n_j \to \vee S^n_i$ is the attaching map for the j-th $(n+1)$-dimensional cell, this means that $X^{(n+1)}$ is the pushout of the diagram

$$\vee_j D^{n+1}_j \leftarrow \vee_j S^n_j \xrightarrow{f = \vee f_i} \vee_i S^n_i$$

2. The Hurewicz homomorphism

We know that $H_n(S^n; \mathbb{Z}) \cong \mathbb{Z}$ and fix a generator u_n for this group by choosing first a specific representative for $u_1 \in H_1(S^1; \mathbb{Z})$. Our model of the circle is the unit circle in $\mathbb{C}$, the unique 0-cell is 1 and the 1-cell given by a map $u \colon \Delta^1 \to S^1$ going around the circle counterclockwise. This map u is a cycle the singular chain complex and it represents u_1. Excision induces isomorphisms $H_1(S^1;\mathbb{Z}) \cong H_n(S^n;\mathbb{Z})$ and we choose u_n to be given by the image of u_1.

DEFINITION 2.1. Let X be a (pointed) path-connected space. The *Hurewicz homomorphism* $Hu\colon \pi_n X \to H_n(X;\mathbb{Z})$ is defined by $Hu(\alpha) = \alpha_*(u_n)$ where a representative $\alpha\colon S^n \to X$ induces a map $\alpha_*\colon H_n(S^n;\mathbb{Z}) \to H_n(X;\mathbb{Z})$.

We have been sloppy about various points here, but they do not matter. First we have not specified the base point, but the change of base points yields isomorphic homotopy groups. Second we have abusively written α for a homotopy class and a representative. Homotopic maps induce the same homomorphism in homology, so different representatives yield the same element $Hu(\alpha)$. Homotopy groups and homology groups are functors, we check next that the Hurewicz map defines a natural transformation.

PROPOSITION 2.2. *The Hurewicz homomorphism is a natural.*

PROOF. Compatibility with the group law comes from the definition of the sum for homotopy groups and the fact that the homology of a wedge is a direct sum by excision. Let $\alpha, \beta\colon S^n \to X$ be two (pointed) maps representing homotopy classes. Their sum $\alpha + \beta$ is given by the composite

$$S^n \xrightarrow{pinch} S^n \vee S^n \xrightarrow{\alpha\vee\beta} X \vee X \xrightarrow{\nabla} X$$

Applying homology gives therefore a composition

$$H_n(S^n) \to H_n(S^n) \oplus H_n(S^n) \to H_n X \oplus H_n X \to H_n X$$

through which our generator u_n goes to (u_n, u_n) first, then $(Hu(\alpha), Hu(\beta))$ and finally the sum $Hu(\alpha) + Hu(\beta)$. This shows Hu is a homomorphism (recall that the direct sum $\oplus$ is the coproduct in abelian groups).

Let $f\colon X \to Y$ be a pointed map. We have to show that the following square is commutative:

$$\begin{array}{ccc} \pi_n X & \xrightarrow{Hu} & H_n(X;\mathbb{Z}) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle H_n(f)} \\ \pi_n Y & \xrightarrow[Hu]{} & H_n(Y;\mathbb{Z}) \end{array}$$

To do so we choose a map $\alpha\colon S^n \to X$ and chase its image through the diagram:

$$H_n(f)[Hu(\alpha)] = H_n(f)[H_n(\alpha)(u_n)] = H_n(f \circ \alpha)(u_n) = Hu(f \circ \alpha)$$

where we used functoriality of homology. □

Our main theorem in this section provides an isomorphism between the first non-trivial homotopy group and the first non-trivial reduced homology group of a simply connected space. We already know this is true for a sphere, but we check now that the Hurewicz homomorphism is this isomorphism. The idea of the proof was already present in Hopf's work. Most of the proof we have already seen in exercises, we write it down here for completeness.

LEMMA 2.3. *Let $n \geq 1$. Then $Hu\colon \pi_n S^n \to H_n(S^n;\mathbb{Z})$ is an isomorphism.*

PROOF. We show that every homotopy class of a map $f\colon S^n \to S^n$ is a sum of maps of degree ± 1. So let us consider an arbitrary map $f\colon S^n \to S^n$. By the PL approximation Lemma 2.1 we change f up to homotopy for a map g which is PL on a polyhedron $K \subset S^n = * \cup e^n$ and there is a non-empty open subset $U \subset e^n$ such that $f_1^{-1}(U) \subset K$. If the image of g does not contain the whole subset U, then, just like in the proof of the cellular approximation Theorem 2.2 we choose a point x in U and not in the image. Hence g factors through $S^n \setminus \{x\}$ which is contractible and so g, and therefore f as well, are nullhomotopic.

Let us assume now that g is surjective on U so it is an isomorphism on all simplices in K. We choose again a point $x \in U$ but this time we pick a neighborhood $x \in V \subset U$, homeomorphic to an open ball and such that $g^{-1}(V)$ consists of finitely many homeomorphic balls V_i containing each one preimage x_i of x. The crucial step is to notice that the collapse map $c\colon S^n \to S^n/(S^n \setminus V)$ is a homotopy equivalence (like collapsing a hemisphere). Thus $c \circ g$ can be used instead of g to conclude the argument.

This map sends $S^n \setminus (\cup V_i)$ to the base point, so it factors through the quotient $S^/(S^n \setminus (\cup V_i)) \simeq \vee S^n_i$. On each V_i we had a homeomorphism given by an invertible matrix of determinant ± 1, so that we have now an induced map $S^n_i \to S^n$ of degree

$d_i = \pm 1$. All together we have replaced f by a homotopic map

$$S^n \to \vee S_i^n \xrightarrow{\vee d_i} \vee S_n \xrightarrow{\nabla} S^n$$

The isomorphism $\pi_n S^n \to \mathbb{Z}$ is given by $[f] \mapsto \sum d_i = d$. Now that we know that both $\pi_n S^n$ and $H_n(S^n;\mathbb{Z})$ are isomorphic to $\mathbb{Z}$ it is sufficient to look at the image of the generator through the Hurewicz homomorphism. We choose our favorite degree one map, namely the identity, and compute $Hu([id_{S^n}]) = (id_{S^n})_*(u_n) = u_n$. This is an isomorphism. □

In algebraic topology we had seen that any degree can be realized by a self map of S^1, by $z \mapsto z^d$, and suspending this map yields a degree d map on higher dimensional spheres. What we have sketched above is that any self map of the sphere is homotopic to such a map. We move next from one sphere to a wedge of spheres. Notice that we increase the lowest possible value of n here! Recall from the topology class that $\pi_1(\vee S^1)$ is a free (non-abelian) group, so this does not work for $n = 1$.

Proposition 2.4. *Let $n \geq 2$. Then $Hu\colon \pi_n(\vee S^n_\alpha) \to H_n(\vee S^n_\alpha;\mathbb{Z}) \cong \oplus \mathbb{Z}_\alpha$ is an isomorphism.*

Proof. In Example 1.3 we have identified the homotopy group of a wedge of spheres and chose explicit generators $\iota_\beta\colon S^n_\beta \hookrightarrow \vee S^n_\alpha$, the wedge summand inclusions. Let us compute the image of $[\iota_\beta]$ through the Hurewicz homomorphism.

$$Hu[\iota_\beta] = (\iota_\beta)_*(u_n) \in H_n(\vee S^n_\alpha;\mathbb{Z}) \cong \oplus \mathbb{Z}_\alpha$$

To obtain an expression in terms of coordinates in this free abelian group we project onto each component. When $\alpha = \beta$ then the composite $S^n_\beta \hookrightarrow \vee S^n_\alpha \to S^n_\alpha$ is the identity, but for $\alpha \neq \beta$ we obtain the constant map. Therefore the element in $\oplus \mathbb{Z}_\alpha$ is $e_\beta = (0, 0, \ldots, 1, 0, \ldots)$ the β-th canonical basis element. Just as in the previous lemma we see that the Hurewicz homomorphism sends generators to generators, it is an isomorphism. □

We are finally ready for the general case.

Theorem 2.5. Hurewicz. *Let $n \geq 2$ and X be an $(n-1)$-connected space. Then $Hu\colon \pi_n X \to H_n(X;\mathbb{Z})$ is an isomorphism.*

PROOF. Since both homotopy groups and homology groups are weak homotopy invariants, we might as well suppose that X is an $(n-1)$-connected CW-complex, hence, by Remark 1.4, that the n-skeleton $X^{(n-1)}$ is reduced to a point, and that the n-skeleton $X^{(n)}$ is a wedge of spheres. Moreover we have shown in Lemma 1.2 that the homotopy and homology groups we are interested in only depends on the $(n+1)$-st skeleton, so we can assume by naturality that $x = X^{(n+1)} = \vee S^n_\alpha \cup (\cup e^{n+1}_\beta)$. Let $f\colon \vee S^n_\beta \to \vee S^n_\alpha$ be the wedge of all attaching maps, so $X = C(f)$ and we call i the inclusion of the n-skeleton. By naturality we have a commutative diagram

$$\begin{array}{ccccc}
\pi_n(\vee S^n_\beta) & \xrightarrow{f_*} & \pi_n(\vee S^n_\alpha) & \xrightarrow{i_*} & \pi_n X \\
{\scriptstyle Hu}\downarrow\cong & & {\scriptstyle Hu}\downarrow\cong & & \downarrow{\scriptstyle Hu_X} \\
H_n(\vee S^n_\beta) & \xrightarrow{f_*} & H_n(\vee S^n_\alpha) & \xrightarrow{i_*} & H_n X \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
C^{cell}_{n+1}(X) & \xrightarrow{\partial} & C^{cell}_n(X) & \twoheadrightarrow & \mathrm{Coker}\partial
\end{array}$$

Proposition 2.4 explains why we have vertical isomorphisms for wedges of spheres, and the bottom identification with the cellular chain complex is clear since the differential is defined by the attaching map. On the right we have identified $H_n X$ with the cokernel of this differential since $C^{cell}_{n-1}(X) = 0$, there are no cells in dimension $(n-1)$. This shows that in homology i_* is surjective, therefore so is the Hurewicz homomorphism Hu_X for X.

We finally have to prove that it also injective. Let us look at the kernel. Let $\omega \in \pi_n X$ be an elemt with $Hu_X(\omega) = 0$. By the cellular approximation Theorem 2.2 we can represent ω by a cellular map $w\colon S^n \to X^{(n)} = \vee S^n_\alpha$. Therefore $0 = Hu_X(\omega) = i_*(Hu([w])$. The middle line in our diagram is exact, it is part of a long exact sequence in homology, so there is yet another homology class $\sigma \in H_n(\vee S^n_\beta)$ with $Hu([w]) = f_*(\sigma)$. We lift σ through the Hurewicz isomorphism to a class $\zeta \in \pi_n(\vee S^n_\beta)$ and observe that $\omega = i_*([w]) = i_*(f_*(\zeta))$. But this is zero because the composition $i \circ f$ is nullhomotopic, it factors through a (contractible) wedge of discs. □

We add an important remark about the case $n = 1$ and an extension of the Hurewicz Theorem to the next degree.

REMARK 2.6. For path-connected spaces the Hurewicz homomorphism $Hu : \pi_1 X \to H_1(X;\mathbb{Z})$ is not an isomorphism in general as illustrated by $S^1 \vee S^1$. One can see for this example that Hu is the quotient of the free group $\mathbb{Z} * \mathbb{Z} \twoheadrightarrow \mathbb{Z} \oplus \mathbb{Z}$. In fact one can show, following the same strategy as in our proof, that Hu is the abelianization of the fundamental group.
When X is $(n-1)$-connected and $n \geq 2$, then we have seen that $Hu : \pi_n X \to H_n(X;\mathbb{Z})$ is an isomorphism. We can then go even further and show that the next Hurewicz homomorphism $Hu : \pi_{n+1} X \to H_{n+1}(X;\mathbb{Z})$ is an epimorphism.

We have managed to construct some special spaces having a fixed homotopy group. For homotopy groups this was not so easy, but for homology groups computations are much easier with the cellular chain complex. Now that we have the Hurewicz Theorem to help us, we can realize any abelian group as π_n of some space.

EXAMPLE 2.7. Let A be an abelian group and choose a presentation as quotient of $\varphi\colon \oplus \mathbb{Z}_\beta \hookrightarrow \oplus \mathbb{Z}_\alpha$. We realize this inclusion as a map of spheres $f\colon \vee S^n_\beta \to \vee S^n_\alpha$ and define $M(A,n) = C(f)$ to be the homotopy cofiber. Its cellular chain complex is reduced to φ so we can compute all homology groups. The only non-trivial group is $H_n(M(A,n);\mathbb{Z}) \cong A$. We call $M(A,n)$ a *Moore space* of type (A,n).

3. Eilenberg-Mac Lane spaces

In the previous Example 2.7 we have constructed a space with trivial homotopy groups in degrees $< n$, it is $(n-1)$-connected, and $\pi_n M(A,n) \cong A$. Therefore, by taking the n-th Postnikov section, see Proposition 4.3, we obtain a space $X = M(A,n)[n]$ with a single non-trivial homotopy group, namely $\pi_n X \cong A$. Such a space deserves a name, because it is a building block from the point of view of homotopy groups, just like spheres are building blocks from the point of view of homology, and moreover, we will see that such spaces play also a central role for (co)homology.

DEFINITION 3.1. An *Eilenberg-Mac Lane space* of type $K(A,n)$ is a space X such that $\pi_k X = 0$ for $k \neq n$ and $\pi_n X \cong A$.

We already know such spaces exist, they are also unique up to weak equivalence, by definition. To put this differently, if we require a $K(A,n)$ space to be a CW-complex, then it is even unique up to homotopy. We will call these space $K(A,n)$. Therefore we will write abusively that $K(\mathbb{Z},1) = S^1$, $K(\mathbb{Z}/2,1) = \mathbb{R}P^\infty$, or $K(\mathbb{Z},2) = \mathbb{C}P^\infty$.

Of course, when $n = 1$ the group A could be an arbitrary group, but for $n \geq 2$ we have to choose an abelian group. From now on we will only work with abelian groups and consider the functor $[-, K(A,n)]$.

PROPOSITION 3.2. *Pointed homotopy classes of maps* $[-, K(A,n)]$ *defines a contravariant functor from* Top_* *to Ab which sends cofibration sequences* $X \hookrightarrow Y \to Y/X$ *to exact sequences.*

PROOF. Since $K(A,n) \simeq \Omega^2 K(A,n+2)$, we have that such pointed homotopy classes are indeed abelian groups. We know from the Puppe sequence Theorem 1.9 that cofibration sequences are sent to exact sequences, using the homotopy equivalence $Y/X \simeq C(X \subset Y)$ by homotopy invariance of homotopy pushouts, see Theorem 6.7. □

Our aim is to compare this functor with ordinary cohomology with coefficients in A. We establish now a few properties which should remind us of characteristic properties of ordinary cohomology. The first one is the *suspension axiom.*

PROPOSITION 3.3. *For any space* X *we have a natural isomorphism* $[X, K(A,n)] \cong [\Sigma X, K(A,n+1)]$.

PROOF. This is a direct consequence of the loop-suspension adjunction. □

The second one is the *wedge axiom.*

PROPOSITION 3.4. *For any index set* I*, we have an isomorphism* $[\vee X_i, K(A,n)] \cong \prod [X_i, K(A,n)]$.

PROOF. We know that mapping spaces convert wedges into products:

$$\mathrm{map}_*(\vee X_i, K(A,n)) \simeq \prod \mathrm{map}_*(X_i, K(A,n))$$

Taking now sets of components components yields the desired isomorphism of sets (and thus groups). □

Even if this is a very easy computation let us record the value of $[-, K(A,n)]$ on spheres.

LEMMA 3.5. *For all $k \neq n$ we have $[S^k, K(A,n)] = 0$ and we have an isomorphism $[S^n, K(A,n)] \cong A$.* □

Our next goal is to compare $[X, K(A,n)]$ with $H^n(X;A)$, so let us recall how cohomology groups with coefficients are computed. Let $C_*(X)$ be the singular or the cellular chain complex. We dualize and consider the cochain complex $\mathrm{Hom}(C_*(X);A)$. Its cohomology groups are $H^n(X;A)$. To be able to compare ordinary cohomology with homotopy classes of maps into Eilenberg-Mac Lane spaces, we will look for a comparison map.

For this we need to identify a special cohomology class.

LEMMA 3.6. *We have an isomorphism $H^n(K(A,n);A) \cong \mathrm{Hom}(A,A)$.*

PROOF. Since we can build a model of $K(A,n)$ from a Moore space $M(A,n)$ by attaching cells of dimension ≥ 2, the cellular chain complex for $K(A,n)$ looks like that of $M(A,n)$ in degrees $\leq n+1$:

$$\oplus\mathbb{Z}_\beta \xrightarrow{d} \oplus\mathbb{Z}_\alpha \to 0$$

where α belongs to an index set of cells of dimension n, and β for $(n+1)$-dimensional cells. The cokernel of d is isomorphic to A since by construction $H_n(M(A,n);\mathbb{Z}) \cong A$. The cochain complex is thus of the form

$$\mathrm{Hom}(\oplus\mathbb{Z}_\alpha, A) \leftarrow \mathrm{Hom}(\oplus\mathbb{Z}_\beta, A) \leftarrow 0$$

The n-th cohomology group is the kernel of this homomorphism $\mathrm{Hom}(d,A)$, but, since Hom is a left exact functor, this is $\mathrm{Hom}(\mathrm{Coker}(d),A) \cong \mathrm{Hom}(A,A)$. □

The identity map $id_A \colon A \to A$ represents an (important) cohomology class ι_A.

DEFINITION 3.7. For any natural number n we define a natural transformation of functors $T\colon [X, K(A,n)] \to H^n(X;A)$ by sending the homotopy class of a map f to $H^n(f;A)(\iota_A)$.

In more details, f induces a map in cohomology $H^n(K(A,n);A) \to H^n(X;A)$ and we use ι_A to push it to the cohomology of X.

PROPOSITION 3.8. *The natural transformation T is an isomorphism on all finite dimensional CW-complexes.*

PROOF. The cellular complex for S^n consists in two copies of $\mathbb{Z}$ in degrees 0 and n, the cochain complex allows us thus to compute easily $H^n(S^n;A) \cong A$. To compare both functors we compute T on homomorphisms $\mathbb{Z} \to A$ corresponding to a generator α and represent it by a map $S^n \to K(A,n)$ factoring through $\vee S^n_\alpha$. Its image is precisely the generator α by construction.

By the wedge axiom, see Proposition 3.4, we also obtain an isomorphism on arbitrary wedges of spheres, and since both functors have long exact sequences associated to cofiber sequences, we obtain by the five Lemma an isomorphism on all finite dimensional CW-complexes by induction on skeleta. Assume T is an isomorphism on $X(k)$ and consider the cofiber sequence $\vee S^k_\alpha \to X^{(k)} \to X^{(k+1)}$. It induces long exact sequences in cohomology and also in $[-, K(A,n)]$, and we conclude that so is $T\colon [X^{(k+1)}, K(A,n)] \to H^n(X^{(k+1)};A)$. □

4. The Milnor sequence

To understand what happens for arbitrary CW-complexes we know how to compute the cohomology of an infinite dimensional CW-complex (in a given degree it only depends on the lower dimensional cells), but we need to be able to compute $[X, K(A,n)]$ for such an infinite dimensional CW-complex X. For this we consider $X = \cup X^{(n)}$ as filtered space. In general if $X_n \subset X$ is an increasing and exhaustive sequence of subspaces of X, we apply cohomology (or homotopy classes into $K(A,n)$) and obtain a tower

$$\dots \to H^n(X_{n+1};A) \to H^n(X_n;A) \to \dots$$

and we have to compare the (inverse) limit of this tower with $H`n(X;A)$. Let us thus do algebra in this section, but do not worry, we will meet again CW-complexes in the next section.

DEFINITION 4.1. Let $(\mathbb{N}, \leq)$ be the poset of natural numbers and we write simply $\mathbb{N}^{op}$ for the opposite poset. A *tower of abelian groups* is a functor $A_\bullet\colon \mathbb{N}^{op} \to Ab$, i.e. a diagram of the form $\dots A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} \dots \to A_1 \to A_0$.

DEFINITION 4.2. Let $A_\bullet\colon \mathbb{N}^{op} \to Ab$ be a tower of abelian groups. The *inverse limit* $\lim A_\bullet$ is the abelian subgroup of $\prod A_n$ consisting of all *compatible sequences* $(a_n)_{n\in\mathbb{N}}$ such that $f_n(a_n) = a_{n-1}$.

REMARK 4.3. As usual we can interpret this explicit description of a limit as an adjunction. For this we consider the *constant tower* functor $c\colon Ab \to Ab^{\mathbb{N}^{op}}$. The limit functor is then a right adjoint, one checks indeed that a morphism from a constant tower $cA \to A_\bullet$ corresponds exactly to a homomorphism $A \to \lim A_\bullet$.

EXAMPLE 4.4. One example that might have appeared in other courses is the inverse limit $\lim \mathbb{Z}/p^n = \mathbb{Z}_p^\wedge$ where each homomorphism in the tower is the reduction mod p^n.

We introduce now a group homomorphism that shifts by one all group elements in the product $\prod A_n$ using the homomorphisms from a tower. Since all these are group homomorphisms, so is the shift map.

DEFINITION 4.5. Let $A_\bullet$ be a tower. The *shift map* $\mathrm{sh}\colon \prod A_n \to \prod A_n$ is defined by $(a_n)_{n\geq 0} \mapsto (f_n(a_n))_{n\geq 1}$. The cokernel of $\mathrm{sh} - id$ is the *first derived functor of the limit*, written $\lim^1 A_\bullet$.

We will see why this deserves to be called a derived functor, but let us first establish a close relationship with the limit.

LEMMA 4.6. *The kernel of* $\mathrm{sh} - id$ *is equal to the limit.*

PROOF. By definition of the shift map the kernel of this difference consists precisely of all sequences (a_n) for which $f_n(a_n) - a_{n-1} = 0$. □

We are ready now for the six term exact sequence, very much like the Hom-Ext sequence you have seen in homological algebra. In the following proposition we use the notion of exactness for morphisms of towers, by which we mean that for each level n we have exactness for abelian groups.

PROPOSITION 4.7. *Let* $0 \to A_\bullet \xrightarrow{\alpha_\bullet} B_\bullet \xrightarrow{\beta_\bullet} A_\bullet \to 0$ *be a short exact sequence of towers of abelian groups. Then we have a six term exact sequence of abelian groups*

$$0 \to \lim A_\bullet \to \lim B_\bullet \to \lim C_\bullet \to \lim^1 A_\bullet \to \lim^1 B_\bullet \to \lim^1 C_\bullet \to 0$$

PROOF. We apply the Snake Lemma to the following diagram

$$\begin{array}{ccccc} \prod A_n & \xrightarrow{\prod \alpha_n} & \prod B_n & \xrightarrow{\prod \beta_n} & \prod C_n \\ \downarrow{\scriptstyle \mathrm{sh}-id} & & \downarrow{\scriptstyle \mathrm{sh}-id} & & \downarrow{\scriptstyle \mathrm{sh}-id} \\ \prod A_n & \xrightarrow{\prod \alpha_n} & \prod B_n & \xrightarrow{\prod \beta_n} & \prod C_n \end{array}$$

Notice that the product of the injective maps α_n is injective (a limit being a right adjoint is left exact) and the likewise the product of surjective maps β_n is surjective, which explains the zeros at the beginning and end of the six term exact sequence. □

LEMMA 4.8. *Let* $A_\bullet$ *be a tower of abelian groups. If there exists an integer* k *such that* $f_n \colon A_n \to A_{n-1}$ *is surjective for all* $n > k$, *then* $\lim^1 A_\bullet = 0$.

PROOF. We apply the definition and show that the shift map minus the identity is surjective. Let $(a_0, a_1, \dots)$ be an arbitrary element in the product $\prod A_n$. We are looking for an element $(b_0, b_1, \dots)$ such that $a_\ell = f_{\ell+1}(b_{\ell+1}) - b_\ell$ for all $\ell \in \mathbb{N}$. Since f_{n+1} is surjective we choose b_{n+1} such that $f_{n+1}(b_{n+1}) = a_n$ and $b_n = 0$. We define by downward induction the b_ℓ's for $0 \leq \ell < n$ starting with $b_{n-1} = -a_{n-1}$, $b_{n-2} = f_{n-1}(b_{n-1}) - a_{n-2}$, etc.

For the higher b_ℓ's we proceed by upward induction and use surjectivity of the f_ℓ's. We have to solve the system of equations given by $a_\ell - b_\ell = f_{\ell+1}(b_{\ell+1})$ for $\ell \geq n+2$, which has a solution since f_ℓ is surjective. □

REMARK 4.9. There is a weaker (hence better) condition ensuring the vanishing of $\lim^1$, called the *Mittag-Leffler condition*. Instead of surjectivity of all but a finite number of f_n's we only require that for each k there is an integer $j \geq k$ such that the images of compositions of structure maps coincide: $\mathrm{Im}(A_i \to A_k) = \mathrm{Im}(A_j \to A_k)$ for all $i \geq j$.

5. Cohomology of arbitrary CW-complexes

After this purely algebraic interlude we come back to cohomology. We had left after we had understood that ordinary cohomology and homotopy classes into Eilengerb-Mac Lane spaces coincide on finite dimensional CW-complexes. Our next aim is to see this also holds for arbitrary CW-complexes. For this we consider a filtered space $X = \cup X_k$ and wish to understand the relationship between the cohomology of X and the inverse limit of the tower $H^n(X_k; A)$. For us the important case is the skeletal filtration of course. We start with a statement without proof, but this should be reminiscent of the construction of homotopy pushouts.

PROPOSITION 5.1. *Let $X_\bullet \colon \mathbb{N} \to Top$ be a diagram of the form $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to \dots$. The homotopy colimit of $X_\bullet$ can be computed as the (strict) colimit of an equivalent diagram where all maps f_n have been replaced by cofibrations.*

REMARK 5.2. The standard model of hocolim$X_\bullet$ is given by the so-called *telescope*. We obtain the equivalent diagram mentioned in the previous proposition by turning each map into a cofibration with our favorite method, namely the cylinder. Thus we replace $f_0 \colon X_0 \to X_1$ by the inclusion $X_0 \hookrightarrow \mathrm{Cyl}(f_0) = (X_0 \times I) \cup X_1$. We think of this inclusion as a horizontal cylinder lying over the positive real numbers and gluing X_1 at the end of the cylinder, i.e. identifying $X_0 \times 1$ with its image in X_1. We continue by turning the composite map $\mathrm{Cyl}(f_0) \to X_1 \xrightarrow{f_1} X_2$ into $hbox\mathrm{Cyl}(f_0) = (X_0 \times [0,1]) \cup X_1 \hookrightarrow (X_0 \times [0,1] \cup (X_1 \times [1,2]) \cup) X_2$ where we attach the space X_2 on the right of the cylinder on X_1, i.e. identifying $X_1 \times 2$ with its image in X_2. We continue inductively obtaining a (long) telescope. The union of these spaces is written $Tel(X_\bullet)$ and provides a model for the homotopy colimit.

Working with pointed spaces, one can replace the cylinders with their pointed version by collapsing an interval. For well-pointed spaces the homotopy type is the same. Therefore, if we start with a pointed diagram of well-pointed spaces, the pointed telescope has the same homotopy type as the unpointed version.

One important consequence of Proposition 5.1 is the homotopy invariance. In particular if our original diagram $X_\bullet$ comes from a filtered space with cofibrations $X_n \subset X_{n+1}$, then one could construct the homotopy colimit as the union $\cup X_n$ since all maps are already cofibrations, but we could also wish to construct the telescope as explained above. Then $Tel(X_\bullet) \simeq X$.

In order to use the algebraic tools developed in Section 4 we have to realize the identity minus the shift map topologically. Let us write ΣX for the reduced suspension (and recall that for a well-pointed space, for example a CW-complex, the reduced suspension and the unpointed suspension have the same homotopy type.

Lemma 5.3. *Let $f\colon X \to X$ be a pointed self map of (X, x_0) and define Y to be the homotopy pushout of the diagram $X \xleftarrow{\nabla} X \vee X \xrightarrow{(id,f)} X$. Then we have a cofibration sequence $\Sigma X \xrightarrow{id-f} \Sigma X \to \Sigma Y$.*

Proof. The homotopy cofiber of the fold map ∇ is the reduced suspension $(X \ltimes I)/(X \vee X)$. Consider thus the following diagram of horizontal cofiber sequences:

$$\begin{array}{ccccc} X \vee X & \hookrightarrow & X \ltimes I & \longrightarrow & \Sigma X \\ {\scriptstyle id} \downarrow {\scriptstyle f} & & \downarrow & & \| \\ X & \hookrightarrow & Y & \longrightarrow & \Sigma X \end{array}$$

Both cofibers are homeomorphic because the left hand side square is a pushout square. The Puppe sequence of the first cofibration allows us to continue two steps further on the right:

$$X \vee X \hookrightarrow X \longrightarrow \Sigma X \xrightarrow{\partial} \Sigma X \vee \Sigma X \xrightarrow{-\Sigma\nabla} \Sigma X$$

We have replaced the reduced cylinder by the homotopy equivalent space X and used the fact that suspension commutes with wedges, both being colimits (use Fubini). But who is ∂? One could think at first sight that it is the pinch map p, but notice that $(-\Sigma\nabla) \circ p$ is not null-homotopic, so it cannot be the pinch map. Identifying carefully the homotopy cofiber construction in the Puppe exact sequence shows that ∂ is in fact $i_1 - i_2$ the difference of the two inclusion maps into the wedge components. Since we work with suspension there is a group structure on pointed homotopy classes and this difference makes sense. We compare now this h-coexact sequence with the one coming from the bottom row.

$$\begin{array}{ccccccc} X & \longrightarrow & \Sigma X & \xrightarrow{i_1 - i_2} & \Sigma X \vee \Sigma X & \xrightarrow{-\nabla} & \Sigma X \\ {\scriptstyle g}\downarrow & & \| & & {\scriptstyle id}\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \Sigma g} \\ Y & \longrightarrow & \Sigma X & \xrightarrow{id - f} & \Sigma X & \longrightarrow & \Sigma Y \end{array}$$

The bottom map $id - f$ has been identified by commutativity of the central square.

□

We apply this result to the shift map sh: $\vee X^{(n)} \to \vee X^{(n)}$ which is defined on the n-skeleton by including it into the $(n+1)$-skeleton.

Corollary 5.4. *Let X be any connected CW-complex. There is a cofibration sequence $\vee \Sigma X^{(n)} \xrightarrow{id - sh} \vee \Sigma X^{(n)} \to \Sigma X$.*

Proof. In order to identify the cofibration sequence from Lemma 5.3 we have to compute the pushout of the following diagram

$$\vee X^{(n)} \xleftarrow{\nabla} (\vee X^{(n)}) \vee (\vee X^{(n)}) \xrightarrow{id \vee sh} \vee X^{(n)}$$

To do this we turn the fold map ∇ into a cofibration by replacing the target by the pointed cylinder $\vee X^{(n)} \rtimes I$. The strict puhsout of this new diagram is then a quotient of this wedge of reduced cylinders $\vee(X^{(n)} \rtimes I)$ where we identify the right hand side $(x_n, 1)$ of the n-th one with $(sh(x_n), 1) = (x_n, 1)$ in the $(n+1)$-skeleton. This is exactly the telescope of the skeletal filtration of X. As observed in Remark 5.2 this is homotopy equivalent to X by homotopy invariance since inclusions of skeleta are cofibrations. □

Proposition 5.5. *Let X be a connected CW-complex. There is an exact sequence*

$$0 \to \lim{}^1 H^{k-1}(X^{(n)}; A) \to H^k(X; A) \to \lim H^k(X^{(n)}; A) \to 0$$

Proof. The above homotopy cofiber sequence induces a long exact sequence in cohomology. We do not write the coefficients in this proof, but A's are understood throughout. Before writing it down we use the suspension axiom to identify $H^{k+1}\Sigma X \cong H^k X$ and the wedge axiom to identify $H^k(\vee X^{(n)})$ with $\prod H^k X^{(n)}$ and the toplogical shift map given by inclusion of one skeleton in the next one induces precisely the shift map in cohomology. Thus we get

$$\dots \xleftarrow{1-sh} \prod H^{k+1}X^{(n)} \xleftarrow{\partial} H^kX \leftarrow \prod H^kX^{(n)} \xleftarrow{1-sh} \prod H^kX^{(n)} \leftarrow \dots$$

The short exact sequence is then given by $\mathrm{Coker}(1-sh) \to H^kX \to \mathrm{Ker}(1-sh)$. □

We arrive finally to our central result in this chapter. Eilenberg-Mac Lane spaces represent ordinary cohomology in the sense that pointed homotopy classes of maps to a $K(A,n)$ are in bijection with $H^n(-;A)$.

THEOREM 5.6. *Let X be a connected CW-complex. Then $H^n(X;A) \cong [X, K(A,n)]$ for any $n \geq 1$.*

PROOF. In our setting the tower $H^k(X^{(n)};A)$ stabilizes for $k > n$ since we have filtered X by skeleta. Therefore $\lim^1$ vanishes and we obtain the desired isomorphism. □

REMARK 5.7. For convenience we have dealt with connected pointed CW-complexes, but the previous result extends of course to non-connected CW-complexes if we allow ourselves to choose a base point in each connected component.

A more tricky point is that this result *does not extend* to arbitrary spaces. Whereas ordinary cohomology is a weak homotopy invariant, homotopy classes into an Eilenberg Mac-Lane space are not so! One can construct spaces having the weak homotopy type of a $K(\mathbb{Z},n)$, together with a CW-approximation map $K(\mathbb{Z},n) \to X$ for which there is no inverse map, even up to weak homotopy, so that the fundamental class in cohomology represented by the identity cannot be given by a map $X \to K(\mathbb{Z},n)$...

CHAPTER 6

More adavanced topics

In this final chapter we indicate a few possible directions for further reading. This first course about homotopy theory and homotopy groups opens doors to other fascinating topics.

1. Postnikov invariants

We have seen that in general homology groups or homotopy groups do not determine the (weak) homotopy of a space. A natural question is then to ask what additional information do we need to reconstruct a space X starting with the knowledge of the homotopy groups. One possible answer is the classification of a certain tower by cohomology classes.

EXAMPLE 1.1. Given an arbitrary group π_1 and abelian groups π_n for $n \geq 2$, there always exist a space X such that $\pi_k X \cong \pi_k$ for all $k \geq 1$. We can take $\prod K(\pi_k, k)$, a product of Eilenberg-Mac Lane spaces, also called a *generalized Eilenberg-Mac Lane space* or *GEM* for short. But there are many others. We have seen for example that $\mathbb{R}P^2$ does not admit $\mathbb{R}P^\infty = K(\mathbb{Z}/2, 1)$ as a retract, so it cannot be a GEM.

To make it simple, let us assume here that X is a simply connected space (there are generalizations to so-called nilpotent spaces, but the theory does not work for arbitrary path connected spaces. We look at the Postnikov tower and recall that we have maps $X \to X[n]$ to the n-th Postnikov section and structure maps forming a tower $p_{n+1}\colon X[n+1] \to X[n]$ of fibrations. This map induces isomorphisms on homotopy groups up to degree n and since $X[n+1]$ has its last possibly non-trivial homotopy group in degree $n+1$, the long exact sequence for the fibration sequence $F \to X[n+1] \xrightarrow{p_{n+1}} X[n]$ shows that F is a $K(\pi_{n+1}X, n+1)$.

The next result is the central point in the theory of Postnikov invariant. In general we know from the dual Puppe sequence that we can extend a fibration sequence on the left and identify the next homotopy fibers as loop spaces on the original spaces. However one cannot *deloop* an arbitrary space and even if one could one would not be able to deloop an arbitrary map between loop spaces. The next proposition should thus arrive as a nice surprise.

PROPOSITION 1.2. *Let* $p\colon X \twoheadrightarrow Y$ *be a fibration of simply connected spaces whose (homotopy) fiber is a* $K(A, n+1)$ *for* $n \geq 2$. *There exists then a map* $k\colon Y \to K(A, n+2)$ *such that* X *is the homotopy fiber of* k.

PROOF. We see in the long exact sequence in homotopy for the fibration sequence $F \to X \to Y$ that the fiber has a single non-trivial homotopy group, and therefore $\pi_{n+1}X \to \pi_{n+1}Y$ is surjective and $\pi_k X \cong \pi_k Y$ for all $k \leq n$.

We turn now the fibration p into a cofibration and consider thus the map as a pair (Y, X). The long exact sequence in homotopy shows that $\pi_k(Y, X) = 0$ for all $k \leq n+1$ since it looks like

$$\cdots \to \pi_{n+1}X \twoheadrightarrow \pi_{n+1}Y \to \pi_{n+1}(Y, X) \xrightarrow{\partial} \pi_n X \xrightarrow{\cong} \pi_n Y \to \pi_n(Y, X) \to \dots$$

We conclude that $\pi_k(Y, X) = 0$ for all $k \leq n+1$. From the relative Hurewicz Theorem we conclude that the homology groups $H_k(Y, X)$ are also all zero for $k \leq n+1$. Moreover

$$H_{n+2}(Y, X) \cong \pi_{n+2}(Y, X) \cong \pi_{n+1}F \cong A$$

We finally look at the cofiber sequence $X \hookrightarrow Y \to Y/X$ and the associated long exact sequence in homology. By an analogous argument as above for homotopy groups we deduce that Y/X is $(n+1)$-connected and the first non-trivial homotopy group if $\pi_{n+2}Y/X \cong H_{n+2}(Y/X, \mathbb{Z}) \cong A$.

We define k to be the composite $Y \to Y/X \to (Y/X)[n{+}2]$. The latter Postnikov section is clearly an Eilenberg-Mac Lane space $K(A, n{+}2)$. We define thus the space X' as the homotopy fiber of k and one can conclude (...) by comparing X with X' and showing they are weakly homotopy equivalent. □

Since cohomology groups are represented by Eilenberg-Mac Lane spaces, we can view the map $k\colon Y \to K(A, n+2)$ as a cohomology class $k \in H^{n+2}(Y; A)$ when Y is a CW-complex.

Definition 1.3. The element $k \in H^{n+2}(Y; A)$ associated to the fiber sequence $K(A, n+1) \to X \to Y$ is the *k-invariant* of this fiber sequence.

An inductive process allows then to reconstruct a simply connected CW-complex from the purely algebraic data given by its homotopy groups $\pi_n X$ and its k-invariants $k_n \in H^{n+2}(X[n]; \pi_{n+1} X)$. We start with $X[2] = K(\pi_2 X, 2)$ and construct $X[3]$ as the homotopy fiber of the k-invariant $K(\pi_2 X, 2) \to K(\pi_3 X, 4)$. We continue inductively and take the (homotopy) limit of the tower of fibrations $\dots X[n+1] \twoheadrightarrow X[n] \twoheadrightarrow X[n-1] \twoheadrightarrow \dots$. This inverse limit is not a CW-complex in general, so we have to take its CW-approximation to get back our original space.

Proposition 1.4. *A space is a GEM if and only if its k-invariants are all zero.*

The if direction is obvious by construction, but the only if needs a proof we do not provide here. We finish this section with an example related to the 3-dimensional sphere.

Example 1.5. Let us consider the Postnikoc section $S^3[3] = K(\mathbb{Z}, 3)$. Who is the next Postnikov section $S^3[4]$?

One can compute $\pi_4 S^3 \cong \mathbb{Z}/2$, generated by the suspension of the Hopf map and it is possible to identify the first k-invariant as the composite map

$$K(\mathbb{Z}, 3) \to K(\mathbb{Z}/2, 3) \to K(\mathbb{Z}/2, 5)$$

where the first map is reduction mod 2 and the second map is much more interesting since it is the so-called *Steenrod square Sq^2*.

2. Poincaré sphere

From here on we will not provide any proof. The short final sections in these notes are only meant to illustrate a few topics which are very closely related to the material we have studied this semester. We start with the famous homology sphere consructed by Poincaré at the beginning of the 20-th century.

There is a simple group of 60 elements, the alternating group A_5 that csan be realized as the group of isometries of the regular icosahedron. It is as such a subgroup of $SO(3)$. The universal cover of $SO(3)$ is the 3-sphere S^3, the unit sphere in the quaternions $\mathbb{H}$ and $\pi_1 SO(2) \cong \mathbb{Z}/2$. Hence, the preimage of A_5 in S^3 is a group of 120 elements often called the icosahedral group I. It is a perfect group, $H_1(K(I,1);\mathbb{Z}) \cong P_{ab} = 0$ and it is even *superperfect*, i.e. $H_2(K(I,1);\mathbb{Z}) = 0$.

Since I is a (discrete) subgroup of the topological group S^3, it acts freely on the right by multiplication and yields a covering space $S^3 \to S^3/I = X$. This space X has fundamental group isomorphic to I by construction, so $H_1(X;\mathbb{Z}) = 0$ and it follows in fact from the fact that I is superperfect that also $H_2(X;\mathbb{Z}) = 0$. It is a three-dimensional CW-complex with a single non-trivial reduced homology group, $H_3(X;\mathbb{Z}) \cong \mathbb{Z}$. This is the famous *Poincaré sphere*, a homology sphere which cannot be homotopy equivalent to S^3 since it has non-trivial fundamental group.

3. James construction and infinite symmetric products

There is a combinatorial model for the "loop-suspension" construction $\Omega\Sigma X$ due to James, which is a powerful tool to understand the Freudenthal suspension Theorem. Let us start with the statement of this important result. If α is a homotopy class in $\pi_k X$ we can represent it by a pointed map $a\colon S^k \to X$ and then suspend it to obtain $\Sigma\alpha\colon \Sigma S^k \to \Sigma X$ representing a class in $\pi_{k+1}\Sigma X$. The suspension operation thus yields a homomorphism $\pi_k X \to \pi_{k+1}\Sigma X$. The Freudenthal suspension Theorem states that this is an isomorphism in a range roughly twice as long as the connectivity of X. More precisely, if X is $(n-1)$-connected for some integer $n \geq 2$, then it is an isomorphism for all $k < 2n-1$.

Back to the homotopy groups of ΣX, we can use the loop-suspension adjunction and identify $\pi_{k+1}\Sigma X$ with $\pi_k \Omega\Sigma X$. The suspension homomorphism is induced by a map of spaces $X \to \Omega\Sigma X$ we describe now. The *James construction* JX is the topological monoid freely generated by X in a sense we will make precise next. The result we will not prove is that $JX \simeq \Omega\Sigma X$. In a monoid we have to express words of arbitrary length in a convenient way. We also need a unit and fix therefore a basepoint $x_0 \in X$ which will be a strict unit for the multiplication.

For words of length one it is easy we just keep a copy of X and call is J_1X. Next we need words of length two and take therefore $X \times X$ whose elements are pairs (x_1, x_2). However, since we wish x_0 to be a unit we have to identify $x_0 \times X$ and $X \times x_0$ with the words of length one. We set J_2X to be the pushout of $X \leftarrow X \vee X \hookrightarrow X \times X$. By construction there is a map $J_1X \hookrightarrow J_2X$ and we continue inductively by adding words of higher length. Then $JX = \cup J_nX$, Notice that $J_2X/J_1X \approx (X \times X)/(X \vee X) = X \wedge X$, which is $(2n-1)$-connected. This is the key to prove that $X \to JX$ is highly connected.

Let us also mention that the James construction or equivalently $\Omega\Sigma$ recognizes H-spaces: A space X admits an H-space structure if and only if the map $X \to JX$ admits a retraction up to homotopy.

In an analogous way one can construct the free abelian monoid on X by identifying not only the elements of the form (x_0, x) and (x, x_0) with x, but any pair (x_1, x_2) with (x_2, x_1). This means we take the quotient of $X \times X$ under the action of the symmetric group S_2 and continue by taking symmetric powers $SP^nX = X^n/S_n$. The colimit of this sequence is called the *infinite symmetric product* $SP^\infty X$. It is a theorem of Dold and Thom that this space is a GEM having the homotopy type of $\prod K(H_nX, n)$. The natural map $X = SP^1X \to SP^\infty X$ recognizes GEMs and on homotopy induces the Hurewicz homomorphism $\pi_nX \to \pi_nSP^\infty X \cong H_nX$.

4. Puppe's Theorem

We have seen the Fubini Theorem for homotopy pushouts and more generally one could prove that homotopy colimits commute with themselves. However homotopy colimit and homotopy limit usually behave badly, but sometimes they do so. This unexpected feature makes the homotopy theory of spaces more special than other homotopy theories. Consider for example a pushout diagram over a fixed base space as below:

$$\begin{array}{ccccc} X_1 & \longleftarrow & X_0 & \longrightarrow & X_2 \\ & \searrow & \downarrow & \swarrow & \\ & & B & & \end{array}$$

Taking the homotopy pushout yields a new map $X \to B$, but how can we understand its homotopy fiber? It has the homotopy type of the pushout of the respective homotopy fibers! This is known as *Puppe's Theorem*. A fun example allows us to reconstruct $K(\mathbb{Z},2)$ without knowing the cellular structure of $\mathbb{C}P^\infty$.

EXAMPLE 4.1. We start from the Hopf fibration $S^1 \to S^3 \xrightarrow{\eta} S^2$. This fibration is classified by a k-invariant $S^2 \to K(\mathbb{Z},2)$ which is nothing but the second Postnikov section (it must induces an isomorphism on π_2 is we want the homotopy fiber to be 2-connected). We now consider the following pushout diagram over $K(\mathbb{Z},2)$:

$$\begin{array}{ccccc} * & \longleftarrow & S^3 & \xrightarrow{\ \eta\ } & S^2 \\ & \searrow & \downarrow{\scriptstyle *} & \swarrow & \\ & & K(\mathbb{Z},2) & & \end{array}$$

Taking the homotopy pushout gives here a map $S^2 \cup e^4 \to K(\mathbb{Z},2)$ and Puppe's Theorem allows us to identify the homotopy fiber as the homotopy pushout of $S^1 \leftarrow S^1 \times S^3 \to S^3$, where we have used some exercices about homotopy fibers of constant maps and recall that $\Omega K(\mathbb{Z},2) \simeq K(\mathbb{Z},1) \simeq S^1$. This homotopy pushout is a join $S^1 * S^3 \simeq \Sigma(S^1 \wedge S^3) \simeq S^5$. This means on the one hand that we got an interesting map $S^5 \to S^2 \cup e^4$ and on the other hand that, since S^5 is 4-connected, the map $S^2 \cup e^4 \to K(\mathbb{Z},2)$ is 4-connected as well. This being said we can iterate the construction and continue with

$$\begin{array}{ccccc} * & \longleftarrow & S^5 & \xrightarrow{\ \eta\ } & S^2 \cup e^4 \\ & \searrow & \downarrow{\scriptstyle *} & \swarrow & \\ & & K(\mathbb{Z},2) & & \end{array}$$

The same argument as above tells us that the process of attaching a 6-dimensional cell to $S^2 \cup e^4$ gives us a new map $S^7 \to S^2 \cup e^4 \cup e^6$, namely the homotopy fiber of the map to $K(\mathbb{Z},2)$ which has therefore an even better connectivity, it's 6-connected. Repeating this process shows that in the limit we obtain a weak equivalence $S^2 \cup e^4 \cup e^6 \cup e^8 \cup \cdots \simeq K(\mathbb{Z},2)$. In particular the homology of $K(\mathbb{Z},2)$ is concentrated in even degrees where we have a copy of the integers $H_{2n}(K(\mathbb{Z},2);\mathbb{Z}) \cong \mathbb{Z})$ for all $n \geq 0$.

Another application of Puppe's Theorem provides generalized Hopf maps for H-spaces. The product map $X \times X \to X$ fits into a commutative diagram

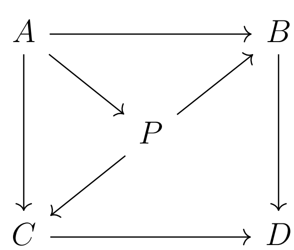
$$\begin{array}{ccccc} X & \xleftarrow{p_1} & X \times X & \xrightarrow{p_2} & X \\ \downarrow & & \downarrow{\scriptstyle m} & & \downarrow \\ * & \longleftarrow & X & \longrightarrow & * \end{array}$$

Taking homotopy pushouts yields a *generalized Hopf map* $X * X \to \Sigma X$. One can check that all vertical homotopy fibers are equivalent to X and most importantly that all horizontal comparison maps between them are equivalences. This implies that the homotopy fiber of the generalized Hopf map is again X, up to homotopy. For the spheres we know to be H-spaces we obtain famous Hopf maps providing non-trivial elements in the homotopy groups of spheres:

(1) For $X = S^1$ we get $S^1 \to S^3 \xrightarrow{\eta} S^2$ and the homotopy cofiber of η is $\mathbb{C}P^2$.
(2) For $X = S^3$ we get $S^3 \to S^7 \xrightarrow{\nu} S^4$ and the homotopy cofiber of ν is $\mathbb{H}P^2$.
(3) For $X = S^7$ we get $S^7 \to S^{15} \xrightarrow{\xi} S^8$ and the homotopy cofiber of ξ is $\mathbb{O}P^2$.

5. Blakers-Massey and Freudenthal Theorems

We finish with the strongest theorem in the above list, probably also the one whose proof is the most difficult. We start from a pushout diagram, forget about the initial object and take the homotopy pullback. By universality there is a comparison map between the initial object and this pullback, the question is what one can say about the homotopy fiber. The situation is illlustrated in the following diagram:



where D is the homotopy pushout of the original pushout diagram and P is the homotopy pullback. There is no formula allowing us in general to identify the homotopy type of $Fib(A \to P)$, but if the pairs (B, A) and (C, A) are respectively m- and n-connected, then the pair (P, A) is $(m + n - 1)$-connected.

As an example we obtain the Freudenthal suspension Theorem.

Example 5.1. The pushout diagram we start with is $* \leftarrow X \rightarrow *$. The homotopy pushout is ΣX and the homotopy pullback is $\Omega\Sigma X$. The comparison map is a map $X \rightarrow \Omega\Sigma X$ which is roughly twice as connected as X itself.

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