Exercise sheet 8

Exercise 5.34. Recall

$$(-L)^\alpha x = \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty t^{\alpha-1}(t-L)^{-1}(-L)x\,dt. \tag{1}$$

As suggested, we split the integral into two parts: $\int_0^\infty = \int_0^K + \int_K^\infty$ for some positive K we are going to choose later. We start with bounding the first integral:

$$\begin{aligned}\left\|\int_0^K t^{\alpha-1}(t-L)^{-1}(-L)x\,dt\right\| &= \left\|\int_0^K t^{\alpha-1}(x-t(t-L)^{-1}x)\,dt\right\| \\ &= \left\|\alpha^{-1}K^\alpha x - \int_0^K t^\alpha (t-L)^{-1}x\,dt\right\| \\ &\le \alpha^{-1}K^\alpha\|x\| + \int_0^K t^\alpha \left\|(t-L)^{-1}\right\|\|x\|\,dt \\ &\le \alpha^{-1}K^\alpha\|x\| + \left\|(-L)^{-1}\right\|(\alpha+1)^{-1}K^{\alpha+1}\|x\|,\end{aligned} \tag{2}$$

where we used the fact that the whole positive complex half-plane belong to the resolvent set of L. This condition also allows us to bound the second integral:

$$\left\|\int_K^\infty t^{\alpha-1}(t-L)^{-1}(-L)x\,dt\right\| \le \int_K^\infty t^{\alpha-2}\|Lx\|\,dt = (1-\alpha)^{-1}K^{\alpha-1}\|Lx\|. \tag{3}$$

Therefore

$$\|(-L)^\alpha x\| \le \pi^{-1}\left(\alpha^{-1}K^\alpha\|x\| + \left\|(-L)^{-1}\right\|(\alpha+1)^{-1}K^{\alpha+1}\|x\| + (1-\alpha)^{-1}K^{\alpha-1}\|Lx\|\right). \tag{4}$$

Take $K = \|Lx\|/\|x\|$. Then we get

$$\begin{aligned}\|(-L)^\alpha x\| &\le \pi^{-1}\left(\alpha^{-1}\|Lx\|^\alpha\|x\|^{1-\alpha} + (\alpha+1)^{-1}\left\|(-L)^{-1}\right\|\|Lx\|^{\alpha+1}\|x\|^{-\alpha} + (1-\alpha)^{-1}\|Lx\|^\alpha\|x\|^{1-\alpha}\right) \\ &\le \pi^{-1}\left(\alpha^{-1} + (\alpha+1)^{-1}\left\|(-L)^{-1}\right\|\sup_{y\in\mathcal{D}(L)}(\|Ly\|/\|y\|) + (1-\alpha)^{-1}\right)\|Lx\|^\alpha\|x\|^{1-\alpha}\end{aligned} \tag{5}$$

since $x \in \mathcal{D}(L)$.

Exercise 5.35. Since B is a bounded operator from $\mathcal{B}_\alpha$ to B, we have $\|Bx\| \le C\|x\|_\alpha = C\|(-L)^\alpha x\|$ for some constant $C > 0$. From the previous exercise, we therefore have

$$\|Bx\| \le C\|Lx\|^\alpha\|x\|^{1-\alpha} \tag{6}$$

for some other $C > 0$. Let us show that the prescribed bound also works. Define

$$f(\epsilon) = \epsilon\|Lx\| + \epsilon^{-\alpha/(1-\alpha)}\|x\|. \tag{7}$$

Let us compute its minimum for $\epsilon \ge 0$. The candidates are $\epsilon = 0$ and the critical point:

$$0 = f'(\epsilon_*) = \|Lx\| - \frac{\alpha}{1-\alpha}\epsilon_*^{-1/(1-\alpha)}\|x\|; \tag{8}$$

$$\epsilon_* = \left(\frac{\alpha}{1-\alpha}\frac{\|x\|}{\|Lx\|}\right)^{1-\alpha}. \tag{9}$$

. Since $f''(\epsilon) \geq 0$ for any $\epsilon \geq 0$, and $\epsilon_* > 0$ the minimum is at the critical point. Therefore for any $\epsilon \geq 0$,

$$f(\epsilon) \geq f(\epsilon_*) = \left(\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\frac{\alpha}{1-\alpha}\right)^{-\alpha}\right)\|Lx\|^{\alpha}\|x\|^{1-\alpha} = \frac{1}{1-\alpha}\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha}\|Lx\|^{\alpha}\|x\|^{1-\alpha} \geq \frac{1}{C}\|Bx\| \tag{10}$$

for some constant $C > 0$ independent on ϵ.

Exercise 5.37. For $\alpha = 0$, commutativity follows trivially. Suppose $\alpha > 0$. Since the domain of $(-L)^{\alpha}$ is the range of $(-L)^{-\alpha}$, we have to check $S(t)(-L)^{\alpha}x = (-L)^{\alpha}S(t)x$ for any $x = (-L)^{-\alpha}y$ for some $y \in \mathcal{B}$:

$$S(t)(-L)^{\alpha}((-L)^{-\alpha}y) = S(t)y; \tag{11}$$

$$\begin{aligned}(-L)^{\alpha}S(t)((-L)^{-\alpha}y) &= (-L)^{\alpha}S(t)\frac{1}{\Gamma(\alpha)}\int_0^{\infty}u^{\alpha-1}S(u)y\,du \\ &= (-L)^{\alpha}\frac{1}{\Gamma(\alpha)}\int_0^{\infty}u^{\alpha-1}S(u)S(t)y\,du \\ &= (-L)^{\alpha}(-L)^{-\alpha}(S(t)y) = S(t)y.\end{aligned} \tag{12}$$

Suppose $\alpha < 0$. We have to check $S(t)(-L)^{\alpha}x = (-L)^{\alpha}S(t)x$ for any $x \in \mathcal{B}$:

$$S(t)(-L)^{\alpha}x = S(t)\frac{1}{\Gamma(-\alpha)}\int_0^{\infty}u^{-\alpha-1}S(u)x\,du = \frac{1}{\Gamma(-\alpha)}\int_0^{\infty}u^{-\alpha-1}S(u)S(t)x\,du = (-L)^{\alpha}S(t)x. \tag{13}$$

Finally, $S(t)$ leaves $\mathcal{B}_{\alpha}$ invariant for any $\alpha > 0$. Indeed, for any $x \in \mathcal{B}_{\alpha} = \text{range}(-L)^{-\alpha}$, $x = (-L)^{-\alpha}y$ for some $y \in \mathcal{B}$. Therefore $S(t)x = S(t)(-L)^{-\alpha}y = (-L)^{-\alpha}S(t)y \in \text{range}(-L)^{-\alpha} = \mathcal{B}_{\alpha}$.

Exercise 5.43.