Exercise sheet 7

Exercise 5.20. First note that since L is negative definite, $f_L(\xi) \leq 0$ for any $\xi \in \mathcal{M}$. Indeed, if $f_L(\xi) > 0$ for some ξ, take $x = K^{-1}g$ for $g = 1_\xi \in L^2(\mathcal{M}, \mu)$ to arrive into a contradiction.

The semigroup property obviously holds; let us check that $t \to e^{Lt}x = K^{-1}e^{f_L t}Kx$ is continuous at $t = 0$ for any $x \in \mathcal{H}$:

$$\lim_{t \to 0+} \|e^{Lt}x - x\| = \lim_{t \to 0+} \|(e^{f_L t} - 1)Kx\| \tag{1}$$

since K is an isomorphism. Since f_L is non-positive, $\xi \to (e^{f_L(\xi)t} - 1)(Kx)(\xi)$ is pointwise bounded by Kx, an integrable function. Therefore we can push the limit inside the norm by dominated convergence and get zero. This implies that e^{Lt} is strongly continuous by Exercise 5.2.

Let us prove that its generator is exactly L. For any $x \in \mathcal{D}(L)$,

$$\lim_{t \to 0+} \|t^{-1}(e^{Lt}x - x) - Lx\| = \lim_{t \to 0+} \|(t^{-1}(e^{f_L t} - 1) - f_L)Kx\|. \tag{2}$$

We can then again push the limit inside the norm by dominated convergence since $\xi \to t^{-1}(e^{f_L(\xi)t} - 1)(Kx)(\xi)$ is pointwise bounded by $f_L Kx$, which is integrable, for any $t > 0$, by non-positivity of f_L. The pointwise limit is zero, hence the above limit also is.

Exercise 5.21. Let L be the generator of S. As it is self-adjoint and negative-definite, we can define $\tilde{S}(\lambda) = e^{L\lambda}$. By Exercise 5.20 and the uniqueness property, $S(t) = \tilde{S}(t)$ for any $t \geq 0$.

Recall f_L must be non-positive for a negative-definite L. Fix any $\phi \in (-\pi/2, \pi/2)$. Let us check that $t \to e^{Lte^{i\phi}}x = K^{-1}e^{f_L te^{i\phi}}Kx$ is continuous at $t = 0$ for any $x \in \mathcal{H}$:

$$\lim_{t \to 0+} \|e^{Lte^{i\phi}}x - x\| = \lim_{t \to 0+} \|(e^{f_L te^{i\phi}} - 1)Kx\| \tag{3}$$

since K is an isomorphism. Since f_L is non-positive and $|\phi| < \pi/2$, $\xi \to (e^{f_L(\xi)te^{i\phi}} - 1)(Kx)(\xi)$ is pointwise bounded by $2Kx$, an integrable function, by absolute value. Therefore we can push the limit inside the norm by dominated convergence and get zero. This implies that $e^{Lte^{i\phi}}$ is strongly continuous for any $\phi \in (-\pi/2, \pi/2)$ by Exercise 5.2.

Also the map $\lambda \to \tilde{S}(\lambda)$ is analytic given $\mathrm{Re}\lambda > 0$ since

$$\lim_{z \to 0} z^{-1}\|e^{L(\lambda+z)} - e^{L\lambda}\| = \lim_{z \to 0} z^{-1}\|e^{f_L(\lambda+z)} - e^{f_L\lambda}\| \leq \sup_{\zeta \in \mathbb{C}:\, \mathrm{Re}\zeta > 0} |e^{f_L\zeta}| \lim_{z \to 0} z^{-1}\|e^{f_L z} - 1\| \leq \lim_{z \to 0} z^{-1}\|e^{f_L z} - 1\|. \tag{4}$$

We again push the limit inside the norm by dominated convergence and get zero. Since the analytic extension is unique, S is analytic on the right half-plane.

Exercise 5.23. Consider a semigroup S on $L^2(\mathbb{R})$ defined as $(S(t)f)(\xi) = f(t + \xi)$. It is strongly continuous by Exercise 5.9. However, $t \to S(t)$ does not admit an analytical continuation to $\{z : |\arg z| < \theta\}$ for any $\theta > 0$. Indeed, if $z \to S(z)$ was analytic in some domain $\mathbb{R}_+ \subset U \subset \mathbb{C}$, $z \to S(z)f$ also would have to be analytic in this domain for any fixed $f \in L^2(\mathbb{R})$. However, if f does not admit an analytic extension from the real line, neither does $z \to S(z)f$.

Exercise 5.29.