Exercise sheet 6

Exercise 5.2. Suppose $(x,t) \to S(t)x$ is continuous. Note that if $\lim_{n\to\infty} \|S(t_n)\| < \infty$ for any $t_n \to 0+$ such that the limit exists then $\|S(t)\| \le Me^{at}$ for some $M, a > 0$ for any $t \ge 0$. Indeed, in this case, there exist $C, \delta > 0$ such that $\sup_{t\in[0,\delta]} \|S(t)\| \le C$. Then any $t \ge 0$ can be expressed as $t = \delta p + q$, where p is a non-negative integer and $q \in [0, \delta)$. Then by the semigroup property, $\|S(t)\| \le \|S(\delta)\|^p \|S(q)\| \le C^{p+1}$, which does not exceed Me^{at} for properly chosen $M, a > 0$.

Therefore for the bound $\|S(t)\| \le Me^{at}$ to fail, we need a sequence $t_n \to 0+$ such that $\lim_{n\to\infty} \|S(t_n)\| = \infty$. If such sequence exists then there exists a sequence $\{x_n\}$ contained in a unit ball such that $\lim_{n\to\infty} \|S(t_n)x_n\| = \infty$. Take $y_n = x_n/\sqrt{\|S(t_n)x_n\|}$; clearly, $y_n \to 0$, while $\lim_{n\to\infty} \|S(t_n)y_n\| = \infty$, which is impossible if $(x,t) \to S(t)x$ is continuous at zero.

In fact, $\|S(t)\| \le Me^{at}$ for any $t \ge 0$ whenever S is a semigroup on $\mathcal{B}$ and $t \to S(t)x$ is continuous at zero for any $x \in \mathcal{B}$. Indeed, if the operator norm bound fails then, as we demonstrated above, there exists a sequence $t_n \to 0+$ such that $\lim_{n\to\infty} \|S(t_n)\| = \infty$. Since $S(t_n)x$ converges to x in $\mathcal{B}$ for any x, $\|S(t_n)x\| \to \|x\|$, and therefore $\sup_n \|S(t_n)x\| < \infty$ for any x. But then $\sup_n \|S(t_n)\| < \infty$ by Banach-Steinhaus theorem, which is a contradiction.

Suppose now $t \to S(t)x$ is continuous at zero for any x and $\|S(t)\| \le Me^{at}$ for any $t \ge 0$. Take some (t,x) and any $\{(t_n, x_n)\}$ converging to (t,x):

$$\|S(t_n)x_n - S(t)x\| \le \|S(t_n)\|\|(x_n - x)\| + \|(S(t_n) - S(t))x\| \le Me^{at_n}\|(x_n - x)\| + \|(S(t_n) - S(t))x\|. \quad (1)$$

The first term converges to zero. We claim that $t \to S(t)x$ is continuous for any x at any t as well, implying that the second term also vanishes.

Let $t_n \to t$ and $s_n = \min(t_n, t)$. Then

$$\|(S(t_n) - S(t))x\| \le \|S(s_n)\|\|(S(t_n - s_n) - S(t - s_n))x\| \le Me^{at}\|(S(|t_n - t|) - S(0))x\| \to 0 \quad (2)$$

for any x. Hence $(t,x) \to S(t)x$ is continuous.

Suppose now $t \to S(t)x$ is continuous at zero for any x only in some dense subset of $\mathcal{B}$, while still $\|S(t)\| \le Me^{at}$ for any $t \ge 0$. Then for any $\epsilon > 0$ and $x \in \mathcal{B}$, take y from the dense subset such that $\|y - x\| < \epsilon$. In this case,

$$\|(S(t_n) - S(t))x\| \le \|(S(t_n) - S(t))y\| + (\|S(t_n)\| + \|S(t)\|)\|y - x\| < \|(S(t_n) - S(t))y\| + M(e^{at_n} + e^{at})\epsilon. \quad (3)$$

Therefore $\limsup_{n\to\infty} \|(S(t_n) - S(t))x\| < 2Me^{at}\epsilon$ for any $\epsilon > 0$, hence $\|(S(t_n) - S(t))x\| \to 0$. As a result, $\|S(t_n)x_n - S(t)x\| \to 0$ as before.

Exercise 5.4. Suppose L is closed. Then by definition, its graph is a closed subset in $\mathcal{B} \times \mathcal{B}$. Therefore if $\{x_n\} \subset \mathcal{D}(L)$ and $\{Lx_n\}$ are both Cauchy sequences in $\mathcal{B}$ then $\{(x_n, Lx_n)\}$ is a Cauchy sequence in $\mathcal{B} \times \mathcal{B}$. Since it lies in the graph and the graph is closed, it converges to a point (x, Lx) in the graph. This implies $\lim x_n = x \in \mathcal{D}(L)$ and $\lim Lx_n = Lx$.

Suppose now for any sequence $\{x_n\} \subset \mathcal{D}(L)$ such that $\{x_n\}$ and $\{Lx_n\}$ are both Cauchy, $\lim x_n \in \mathcal{D}(L)$ and $\lim Lx_n = L \lim x_n$. Note that for any sequence $\{x_n\} \subset \mathcal{D}(L)$ such that $\{(x_n, Lx_n)\}$ is Cauchy, $\{x_n\}$ and $\{Lx_n\}$ are both Cauchy as well. Therefore any Cauchy sequence in the graph has a limit in the graph, meaning that the graph is closed.

Exercise 5.5. For $\ell \in \mathcal{D}(L^*)$, let $\ell' \in \mathcal{B}^*$ be such that $\ell'(x) = \ell(Lx)$ for any $x \in \mathcal{D}(L)$. Suppose $\{\ell_n\} \subset \mathcal{D}(L^*)$ is a Cauchy sequence in $\mathcal{B}^*$ such that $\{\ell'_n\}$ is also a Cauchy sequence in $\mathcal{B}^*$. Let $\ell = \lim \ell_n \in \mathcal{B}^*$ and $\ell' = \lim \ell'_n \in \mathcal{B}^*$. Then for any $x \in \mathcal{D}(L)$, $\ell(Lx) = \lim \ell_n(Lx) = \lim \ell'_n(x) = \ell'(x)$. Since $\mathcal{D}(L)$ is dense in $\mathcal{B}$ and ℓ' has to be continuous, $\ell(Lx) = \ell'(x)$ for any $x \in \mathcal{B}$. Therefore $\ell \in \mathcal{D}(L^*)$.

Exercise 5.9. Consider a semigroup $S(t)$ on $L^2(\mathbb{R})$ given by $(S(t)f)(\xi) = f(\xi + t)$. Clearly, $\|S(t)\| = 1$. Let us show that $G_f(t) := S(t)f$ is continuous for any $f \in C_c(\mathbb{R})$. Since $C_c(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, it will imply that $S(t)$ is strongly continuous by Exercise 5.2.

Indeed, for any $t \geq 0$,

$$\|G_f(t) - f\|_{L^2}^2 = \int_{\mathbb{R}} (f(\xi + t) - f(\xi))^2 \, d\xi. \tag{4}$$

Since f is continuous and compactly supported, we can put $\lim_{t\to 0+}$ into the integral by dominated convergence, thus getting $\lim_{t\to 0+} \|G_f(t) - f\|_{L^2} = 0$.

Let us compute its generator:

$$(Lf)(\xi) = \lim_{t\to 0+} t^{-1}(f(\xi + t) - f(\xi)). \tag{5}$$

The pointwise limit, when exists, is clearly f', which lies in L^2 iff $f \in H^1$. In other words, $\mathcal{D}(L) = H^1$ and $L = \partial_\xi$.

Consider now the heat semigroup. Let us bound its norm:

$$\|S(t)f\|_{L^2}^2 = \frac{1}{4\pi t} \iiint e^{-\frac{|\xi-\eta|^2}{4t}} e^{-\frac{|\xi-\eta'|^2}{4t}} f(\eta) f(\eta') \, d\eta \, d\eta' \, d\xi; \tag{6}$$

$$\int e^{-\frac{|\xi-\eta|^2}{4t}} e^{-\frac{|\xi-\eta'|^2}{4t}} \, d\xi = \int e^{-\frac{2|\xi-(\eta+\eta')/2|^2 + (\eta-\eta')^2/2}{4t}} \, d\xi = \sqrt{2\pi t} e^{-\frac{(\eta-\eta')^2}{8t}}; \tag{7}$$

$$\|S(t)f\|_{L^2}^2 = \frac{1}{\sqrt{8\pi t}} \iint e^{-\frac{(\eta-\eta')^2}{8t}} f(\eta) f(\eta') \, d\eta \, d\eta' = \frac{1}{\sqrt{8\pi t}} \iint e^{-\frac{r^2}{8t}} f(\eta) f(\eta + r) \, d\eta \, dr. \tag{8}$$

By Cauchy-Schwarz,

$$\int f(\eta) f(\eta + r) \, d\eta \leq \|f\|_{L^2}^2. \tag{9}$$

Therefore

$$\|S(t)f\|_{L^2}^2 \leq \frac{\|f\|_{L^2}^2}{\sqrt{8\pi t}} \iint e^{-\frac{r^2}{8t}} \, dr = \|f\|_{L^2}^2, \tag{10}$$

which gives the bound $\|S(t)\| \leq 1$.

Take $f \in C_c^1(\mathbb{R})$ and let us check that $t \to S(t)f$ is continuous at zero. Let $L_f < \infty$ be the Lipschitz contsant of f. We have:

$$\begin{aligned} \|S(t)f - f\|_{L^2}^2 &= \frac{1}{4\pi t} \iiint e^{-\frac{|\xi-\eta|^2}{4t}} e^{-\frac{|\xi-\eta'|^2}{4t}} (f(\eta) - f(\xi))(f(\eta') - f(\xi)) \, d\eta \, d\eta' \, d\xi \\ &\leq \frac{L_f^2}{4\pi t} \mathcal{L}(\operatorname{supp} f) \sup_{\xi \in \operatorname{supp} f} \iint e^{-\frac{|\xi-\eta|^2}{4t}} e^{-\frac{|\xi-\eta'|^2}{4t}} |\xi - \eta||\xi - \eta'| \, d\eta \, d\eta' \\ &= \frac{4t}{\pi} L_f^2 \mathcal{L}(\operatorname{supp} f) \to 0, \end{aligned} \tag{11}$$

where $\mathcal{L}(A)$ is Lebesgue measure of a measurable set A. Since $C_c^1(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, Exercise 5.2 implies that S is strongly continuous.

Let us now compute its generator:

$$(Lf)(\xi) = \lim_{t\to 0+} \frac{1}{\sqrt{4\pi t^3}} \int_{\mathbb{R}} e^{-\frac{|\xi-\eta|^2}{4t}} (f(\eta) - f(\xi)) \, d\eta = \lim_{t\to 0+} \frac{1}{\sqrt{4\pi t^3}} \int_{-\delta}^{\delta} e^{-\frac{r^2}{4t}} (f(\xi + r) - f(\xi)) \, dr \tag{12}$$

for any $\delta > 0$. If f is twice-differentiable at ξ then we can use the Taylor's theorem:

$$(Lf)(\xi) = \lim_{t\to 0+} \frac{1}{\sqrt{4\pi t^3}} \int_{-\delta}^{\delta} e^{-\frac{r^2}{4t}} \left(f'(\xi) r + \frac{1}{2}(f''(\xi) + o_{r\to 0}(1)) r^2 \right) \, dr. \tag{13}$$

The first term integrates to zero by symmetry. Take an arbitrary $\epsilon > 0$ and choose $\delta > 0$ in such a way that the o-term is bounded by ϵ. Then it integrates to $O_{\epsilon\to 0}(\epsilon)$:

$$(Lf)(\xi) = O_{\epsilon\to 0}(\epsilon) + \lim_{t\to 0+} \frac{1}{\sqrt{4\pi t^3}} \int_{-\delta(\epsilon)}^{\delta(\epsilon)} e^{-\frac{r^2}{4t}} \frac{r^2}{2} f''(\xi) \, dr = \lim_{t\to 0+} \frac{1}{\sqrt{4\pi t^3}} \int_{\mathbb{R}} e^{-\frac{r^2}{4t}} \frac{r^2}{2} f''(\xi) \, dr = f''(\xi). \tag{14}$$

Therefore whenever the pointwise limit exists, Lf should be equal to f''. In order for the limit to exist in $L^2(\mathbb{R})$, one needs $f \in H^2$.