Exercise sheet 5

Exercise 4.46.

Exercise 4.50. Suppose $h \in \operatorname{range} A$, i.e. $h = Au$ for some $u \in \mathcal{H}_\mu$. Then

$$\begin{aligned} \|h\|_\nu &= \sup_{\ell \in \mathcal{B}_2^*} \{\ell(h) : C_\nu(\ell, \ell) \le 1\} \\ &= \sup_{\ell \in \mathcal{B}_2^*} \{\ell(Au) : C_\mu(A^*\ell, A^*\ell) \le 1\} \\ &= \sup_{\ell \in \mathcal{B}_2^*} \{(A^*\ell)u : C_\mu(A^*\ell, A^*\ell) \le 1\} \\ &\le \sup_{k \in \mathcal{H}_\mu^*} \{ku : C_\mu(k, k) \le 1\} \\ &= \sup_{v \in \mathcal{H}_\mu} \{\langle v, u\rangle_\mu : \|v\|_\mu \le 1\} = \|u\|_\mu. \end{aligned} \tag{1}$$

Therefore $\|h\|_\nu \le \inf_{u \in \mathcal{H}_\mu} \{\|u\|_\mu : Au = h\} < \infty$ and $h \in \mathcal{H}_\nu$.

Exercise 4.57.

Exercise 4.64. Since A is a bounded linear operator and μ is a Gaussian measure on L^2, $A^*\mu$ is a Gaussian measure on $H_0^{1,2}$. Since $\mathcal{H}$ is $\mathbb{R}$ in our case, $\mathcal{H}'$ is also $\mathbb{R}$. Then by definition, $A\xi(t)$ is a Wiener process if $\mathbb{E}\, A\xi(t)A\xi(s) = s \wedge t$ for any $t, s \in \mathbb{R}$:

$$\mathbb{E}\, A\xi(t)A\xi(s) = \mathbb{E}\, \langle \xi, I_{[0,t]}\rangle\langle \xi, I_{[0,s]}\rangle = \langle I_{[0,t]}, I_{[0,s]}\rangle = s \wedge t. \tag{2}$$

Therefore $A^*\mu$ is a Wiener measure on $H_0^{1,2}$.

Exercise 4.65. By definition, ξ can be realized as a measure in H^{-s} iff $\mathcal{F}^{-1}\left[(1+|t|^2)^{-s/2}\mathcal{F}\xi\right] \in L^2$. We have

$$(1+|\omega|^2)^{-s/2}[\mathcal{F}\xi](\omega) = (1+|\omega|^2)^{-s/2}\mathbb{E}\, e^{-i\omega^T\xi} = (1+|\omega|^2)^{-s/2}; \tag{3}$$

$$\mathcal{F}^{-1}\left[(1+|\omega|^2)^{-s/2}\mathcal{F}\xi\right](x) = \frac{1}{2\pi}\int_{\mathbb{R}^d} e^{i\omega^T x}(1+|\omega|^2)^{-s/2}\, d\omega; \tag{4}$$

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\mathcal{F}^{-1}\left[(1+|\omega|^2)^{-s/2}\mathcal{F}\xi\right](x)\right)^2 dx &= \frac{1}{4\pi^2}\int_{\mathbb{R}^d}\iint_{\mathbb{R}^d\times\mathbb{R}^d} e^{i(\omega-\omega')^T x}(1+|\omega|^2)^{-s/2}(1+|\omega'|^2)^{-s/2}\, d\omega d\omega' dx \\ &= \frac{1}{4\pi^2}\int_{\mathbb{R}^d}(1+|\omega|^2)^{-s}\, d\omega, \end{aligned} \tag{5}$$

which is finite iff $s > d/2$.