Exercise sheet 4

Exercise 4.22. Define $E = \limsup_{\epsilon \to 0} \mathcal{H}_\epsilon(\mathcal{X})/|\log \epsilon|$; we know that $E < \infty$. Let $\{\epsilon_m\}$ be a sequence of positive numbers converging to zero such that $\lim_{m\to\infty} \mathcal{H}_{\epsilon_m}(\mathcal{X})/|\log \epsilon_m| \leq E$. W.l.o.g. suppose $\epsilon_m \leq 2^{-m}$ and $\mathcal{H}_{\epsilon_m}(\mathcal{X}) \leq (E+1)|\log \epsilon_m|$ $\forall m$, or equivalently, $e^{\mathcal{H}_{\epsilon_m}(\mathcal{X})} \leq 2^{m(E+1)}$ $\forall m$.

Define $N_m = e^{\mathcal{H}_{\epsilon_m}(\mathcal{X})}$ and let $\mathcal{N}_m = (x_1^{(m)}, \dots x_{N_m}^{(m)})$ be the corresponding ϵ_m-net. Note that $\mathcal{D} = \cup_{m=1}^\infty \mathcal{N}_m$ is a countable dense subset of $\mathcal{X}$. Let $\beta \in (0, \alpha)$; defining Ω_β the same way as for $\mathcal{X} = [0,1]^d$, we get that the theorem statement follows if $\mathbb{E}\, M_\beta(X) < \infty$, where $M_\beta(X) = \sup_{x \neq y: x,y \in \mathcal{D}} \frac{|X(x)-X(y)|}{d^\beta(x,y)}$.

Let Δ_m be a set of point pairs (x,y), where $x \in \mathcal{N}_m$ and $y \in \mathcal{N}_{m+1}$, such that $d(x,y) \in (\epsilon_m, 2\epsilon_m)$; note that $|\Delta_m| \leq N_m N_{m+1} \leq N_{m+1}^2$. Define $K_m(X) = \sup_{(x,y)\in\Delta_m} |X(x) - X(y)|$. Similarly to the $\mathcal{X} = [0,1]^d$ case, we have for any fixed $\beta' \in (\beta, \alpha)$,

$$\mathbb{E}\, K_m^p(X) \leq C_p \sum_{(x,y)\in\Delta_m} (C(x,x) + C(y,y) - 2C(x,y))^{p/2} \leq \hat{C}_p N_{m+1}^2 \epsilon_m^{\alpha p} \leq \hat{C}_p 2^{(m+1)(E+1)} 2^{-m\alpha p}. \tag{1}$$

Take $p = \frac{2E+2}{\alpha - \beta'}$. Then $\mathbb{E}\, K_m^p(X) \leq \hat{C}_p 2^{-m\beta' p}$ and Jensen's inequality gives $\mathbb{E}\, K_m(X) \leq K 2^{-m\beta'}$ uniformly in m for some constant K.

Given $x, y \in \mathcal{X}$, let m_0 be the maximal m such that $d(x,y) < 2^{-m}$. Let x_{m_0-1} be a point from $\mathcal{N}_{m_0-1}$ such that $d(x, x_{m_0-1}) < 2^{-m_0+1}$ and let y_{m_0} be a point from $\mathcal{N}_{m_0}$ such that $d(x, x_{m_0}) < 2^{-m_0}$. In this case, $d(x_{m_0-1}, y_{m_0}) < 2^{-m_0+2} = 2\epsilon_{m_0-1}$; therefore $(x_{m_0-1}, y_{m_0}) \in \Delta_{m_0-1}$.

For any $n \geq m_0$, let x_n be a point from $\mathcal{N}_n$ such that $d(x, x_n) < 2^{-n}$, and for any $n > m_0$, let y_n be a point from $\mathcal{N}_n$ such that $d(y, y_n) < 2^{-n}$. Then

$$\begin{aligned} |X(x) - X(y)| &\leq |X(x_{m_0-1}) - X(y_{m_0})| + \sum_{n=m_0-1}^\infty |X(x_{n+1}) - X(x_n)| + \sum_{n=m_0}^\infty |X(y_{n+1}) - X(y_n)| \\ &\leq K_{m_0-1} + \sum_{n=m_0-1}^\infty K_n + \sum_{n=m_0}^\infty K_n = 2\sum_{n=m_0-1}^\infty K_n. \end{aligned} \tag{2}$$

By definition of $M_\beta(X)$,

$$\begin{aligned} M_\beta(X) &= \sup_{(x,y)\in\mathcal{D}} \frac{|X(x)-X(y)|}{d^\beta(x,y)} = \sup_{m_0 \geq 1} \sup_{(x,y)\in\mathcal{N}_{m_0}} \frac{|X(x)-X(y)|}{d^\beta(x,y)} \\ &\leq 2 \sup_{m_0\geq 1} \sum_{n=m_0-1}^\infty K_n 2^{\beta m_0} \leq 2 \sup_{m_0 \geq 1} \sum_{n=m_0-1}^\infty K_n 2^{\beta(n+1)} = 2\sum_{n=0}^\infty K_n 2^{\beta(n+1)}. \end{aligned} \tag{3}$$

Taking expectation,

$$\mathbb{E}\, M_\beta(X) \leq 2\sum_{n=0}^\infty 2^{-n\beta'} 2^{\beta(n+1)}, \tag{4}$$

which converges since $\beta' > \beta$.

Exercise 4.23. For any finite set of points $(x_1, \dots, x_N) \subset [0,1]^d$, construct a linear operator $K_{x_{1:N}} : \mathcal{H}^{\oplus N} \to \mathcal{H}^{\oplus N}$ as follows:

$$\langle (h_1, \dots, h_N), K_{x_{1:N}}(k_1, \dots, k_N) \rangle = \sum_{i,j=1}^N \langle h_i, C(x_i, x_j) k_j \rangle. \tag{5}$$

Since $C(x,y)$ is symmetric and positive definite for any $x,y \in [0,1]^d$, $K_{x_{1:N}}$ posesses the same properties. It is also trace class:

$$\begin{aligned}
\operatorname{tr} K_{x_{1:N}} &= \sum_{n=1}^{\infty}\sum_{p=1}^{N}\langle e_n^{(p)}, K e_n^{(p)}\rangle = \sum_{n=1}^{\infty}\sum_{p=1}^{N}\sum_{i,j=1}^{N}\langle e_n, C(x_i,x_j)e_n\rangle 1_{i=p}1_{j=p} \\
&= \sum_{n=1}^{\infty}\sum_{p=1}^{N}\langle e_n, C(x_p,x_p)e_n\rangle = \sum_{p=1}^{N}\sum_{n=1}^{\infty}\langle e_n, C(x_p,x_p)e_n\rangle \\
&= \sum_{p=1}^{N}\operatorname{tr} C(x_p,x_p) = \sum_{p=1}^{N}\int_{\mathcal{H}}\|x\|^2\,\mu_p(dx) = \int_{\mathcal{H}^{\oplus N}}\|(x_1,\dots,x_N)\|^2\,\mu_1(dx_1),\dots,\mu_N(dx_N),
\end{aligned} \tag{6}$$

where μ_p is a Gaussian measure on $\mathcal{H}$ defined by $C(x_p,x_p)$, $\{e_n\}$ is an orthornormal basis in $\mathcal{H}$, and $e_n^{(p)} = (0,\dots,0,e_n,0,\dots,0) \in \mathcal{H}^{\oplus N}$ for e_n staying at position p.

Proposition 4.17 tells us that there is a Gaussian measure $\mu_{x_{1:N}}$ on $\mathcal{H}^{\oplus N}$ with $\hat{C}_{\mu_{x_{1:N}}} = K_{x_{1:N}}$. By Kolmogorov's extension theorem, we can construct a measure μ_0 on $\mathcal{X} = \mathcal{H}^{[0,1]^d}$ endowed with a product σ-algebra such that all finite-dimensional marginals are Gaussian and satisfy

$$\int_{\mathcal{X}}\langle h, f(x)\rangle\langle k, f(y)\rangle\,\mu_0(df) = \langle h, C(x,y)k\rangle \tag{7}$$

for any $x,y \in [0,1]^d$ and any $h,k \in \mathcal{H}$.

Take $\beta \in (0,\alpha)$ and proceed similarly to the case $\mathcal{H} = \mathbb{R}$. A natural way to define $K_m(X)$ is $\sup_{(x,y)\in\Delta_m}\|X(x) - X(y)\|$. Then noting $\mathbb{E}\left(\|X(x)-X(y)\|^2\right) = \operatorname{tr} C(x,x) + \operatorname{tr} C(y,y) - 2\operatorname{tr} C(x,y)$ since $C(x,y)$ is trace class for any $x,y \in [0,1]^d$, the rest of the proof goes through.

Exercise 4.29. If $h \in \mathcal{B}$ lies in $\mathring{\mathcal{H}}_\mu$ then there should exist $h^* \in \mathcal{B}^*$ such that for any $\ell \in \mathcal{B}^*$,

$$\int h(x)\,d\ell(x) = C_\mu(h^*,\ell). \tag{8}$$

In particular, for $\ell = \delta_t$,

$$h(t) = C_\mu(h^*,\delta_t) = \int h^*(f)\delta_t(f)\,d\mu(f) = \int\left(\int f(x)\,dh^*(x)\right)f(t)\,d\mu(f) = \int x \wedge t\,dh^*(x). \tag{9}$$

It is easy to see that this h satisfies the above condition for any $\ell \in \mathcal{B}^*$.

We see that $h(0) = 0$, h has a derivative at t equal to $\dot{h}(t) = h^*([t,1])$ whenever $h^*(\{t\}) = 0$. Since h^* is a measure of bounded variation, h has to be differentiable almost everywhere and $\dot{h}$ should be of bounded variation. It is easy to see that h has to be of bounded variation as well; indeed, its variation is given by

$$\begin{aligned}
V_0^1(h) &= \sup_P \sum_{i=1}^{|P|-1}|h(t_{i+1}) - h(t_i)| \\
&= \sup_P \sum_{i=1}^{|P|-1}\left|\int_{t_i}^{t_{i+1}}(x - t_i)\,dh^*(x)\right| \\
&\le \sup_P \sum_{i=1}^{|P|-1}(t_{i+1}-t_i)V_{t_i}^{t_{i+1}}(h^*) \\
&\le \sup_P \sum_{i=1}^{|P|-1}V_{t_i}^{t_{i+1}}(h^*) \\
&= V_0^1(h^*),
\end{aligned} \tag{10}$$

which is finite since h^* is a measure of bounded variation. Therefore $\mathring{\mathcal{H}}_\mu \subset BV_0([0,1])$, a set of bounded variation functions with $h(0) = 0$.

Since $BV([0,1])$ is dense in $H^{1,2}([0,1])$, $BV_0([0,1])$ is dense in $H^{1,2}_0([0,1])$. Since $H^{1,2}_0([0,1])$ is complete, $\mathcal{H}_\mu \subset H^{1,2}_0([0,1])$.

In fact, for any $h \in C^\infty([0,1])$ such as $h(0) = 0$ (we denote this space by $C_0^\infty([0,1])$ below), there is $h^* \in \mathcal{B}^*$ satisfying the above condition: it is a measure with density $-\ddot{h}$ (such a measure always has bounded variation). Therefore $C_0^\infty([0,1]) \subset \mathring{\mathcal{H}}_\mu$.

Suppose $h \in C_0^\infty([0,1])$. Then h^* has density (wrt the Lebesgue measure) given by $-\ddot{h}(t)$. It allows us to compute the norm of h:

$$\|h\|_\mu = C_\mu(h^*, h^*) = h^*(h) = \int_0^1 h(t)\, dh^*(t) = -\int_0^1 h(t)\ddot{h}(t)\, dt = \int_0^1 \dot{h}^2(t)\, dt, \tag{11}$$

where we applied integration by parts. Since h is smooth, this integrale is finite, therefore $C_0^\infty([0,1]) \subset H^{1,2}_0([0,1])$. Since $C_0^\infty([0,1])$ is dense in $H^{1,2}_0([0,1])$ wrt $\|\cdot\|_\mu$ and $H^{1,2}_0([0,1])$ is complete, its closure is exactly $H^{1,2}_0([0,1])$ and therefore $H^{1,2}_0([0,1]) \subset \mathcal{H}_\mu$.

Exercise 4.30. $h \in \mathbb{R}^n$ lies in $\mathring{\mathcal{H}}_\mu$ iff there exists $h^* \in \mathbb{R}^n$ such that for any $\ell \in \mathbb{R}^n$,

$$h^T \ell = \int (h^{*,T} x)(\ell^T x)\, d\mu(x) = h^{*,T} \left(\int x x^T\, d\mu(x) \right) \ell = h^{*,T} C \ell, \tag{12}$$

where $C = \int x x^T\, d\mu(x)$ is the covariance matrix. The above assertion holds iff h lies in the range of C. Since the range of a linear opeartor in a finite-dimensional space is always complete, $\mathring{\mathcal{H}}_\mu = \mathcal{H}_\mu$. The corresponding h^* is given by $h^* = C^+ h$, where C^+ is a Moore-Penrose pseudo-inverse, and the corresponding norm is $\|h\|_\mu^2 = h^T C^+ C C^+ h = h^T C^+ h$.

Exercise 4.37. $h \in \mathcal{H}$ lies in $\mathring{\mathcal{H}}_\mu$ iff there exists $h^* \in \mathcal{H}$ such that for any $\ell \in \mathcal{H}$,

$$\langle h, \ell \rangle = \int \langle h^*, x \rangle \langle \ell, x \rangle\, d\mu(x) = \langle h^*, K\ell \rangle. \tag{13}$$

Taking ℓ to be e_n for some n, we get $\langle h, e_n \rangle = \langle h^*, \lambda_n e_n \rangle = \langle \lambda_n h^*, e_n \rangle$. Since this should hold for any n and $\{e_n\}$ is a basis, we get $h = \sum_{n=1}^\infty \langle h, e_n \rangle e_n = \sum_{n=1}^\infty \langle \lambda_n h^*, e_n \rangle e_n = K h^*$. Hence h lies in $\mathring{\mathcal{H}}_\mu$ iff it lies in the range of K. Similarly, we get $h^* = \sum_{n=1}^\infty \langle h^*, e_n \rangle e_n = \sum_{n=1}^\infty \langle \lambda_n^{-1} h, e_n \rangle e_n = K^{-1} h$ since $\lambda_n > 0$ for all $n \geq 1$. Therefore $\|h\|_\mu^2 = \langle h^*, K h^* \rangle = \langle h, K^{-1} h \rangle = \|K^{-1/2} h\|$.

By Proposition 4.32, $\mathcal{H}_\mu \subset \mathcal{H}$. Suppose $\{h_m \in \mathring{\mathcal{H}}_\mu\}_{m=1}^\infty$ converges in $\|\cdot\|_\mu$ to some $h \in \mathcal{H}$. Then for any $m \geq 1$,

$$\begin{aligned}
\sum_{n=1}^\infty \lambda_n^{-1} \langle h, e_n \rangle^2 &\leq 2 \sum_{n=1}^\infty \lambda_n^{-1} \langle h - h_m, e_n \rangle^2 + 2 \sum_{n=1}^\infty \lambda_n^{-1} \langle h_m, e_n \rangle^2 \\
&\leq 2\|h - h_m\|_\mu^2 + 2\|h_m\|_\mu^2 \\
&= 2\|h - h_m\|_\mu^2 + 2 \sum_{n=1}^\infty \lambda_n \langle h_m^*, e_n \rangle^2 \\
&\leq 2\|h - h_m\|_\mu^2 + 2 \sup_{n \geq 1} \langle h_m^*, e_n \rangle^2 \sum_{n=1}^\infty \lambda_n \\
&\leq 2\|h - h_m\|_\mu^2 + 2\|h_m^*\|_2^2 \sum_{n=1}^\infty \lambda_n.
\end{aligned} \tag{14}$$

Take any m such that $\|h - h_m\|_\mu^2 \leq 1$. Since $h_m^* \in \mathcal{H}$ and $\sum_{n=1}^\infty \lambda_n < \infty$, the whole bound is finite.

Conversely, if $h \in \mathcal{H}$ such that $\sum_{n=1}^\infty \lambda_n^{-1} \langle h, e_n \rangle^2 < \infty$ then $h^* = K^{-1} h \in \mathcal{R}_\mu$. Since $\mathcal{R}_\mu$ is a closure of $\mathcal{H}^* \simeq \mathcal{H}$ in $L^2(\mathcal{H}, \mu)$ and $L^2(\mathcal{H}, \mu)$ is complete, there is a sequence $\{h_m^* \in \mathcal{H}^*\}_{m=1}^\infty$ converging to h^* in $\|\cdot\|_{L^2(\mathcal{H},\mu)}$. Then the sequence $\{h_m = K h_m^*\}_{m=1}^\infty$ converges to h in $\|\cdot\|_\mu$:

$$\|h_m - h\|_\mu = \langle h_m - h, K^{-1}(h_m - h) \rangle = \langle K(h_m^* - h^*), h_m^* - h^* \rangle = \int \langle h_m^* - h^*, x \rangle^2\, d\mu(x) \to 0. \tag{15}$$

Since all h_m lie in the range of K, $h_m \in \mathring{\mathcal{H}}_\mu$ for all $m \geq 1$. Therefore $h \in \mathcal{H}_\mu$. Finally, the scalar product in $\mathcal{H}_\mu$ can be expressed in terms of norms:

$$
\begin{aligned}
\langle h, k\rangle_\mu &= \frac{1}{2}\left(\|h+k\|_\mu^2 - \|h\|_\mu^2 - \|k\|_\mu^2\right) \\
&= \frac{1}{2}\left(\|K^{-1/2}(h+k)\|^2 - \|K^{-1/2}h\|^2 - \|K^{-1/2}k\|^2\right) \\
&= \langle K^{-1/2}h, K^{-1/2}k\rangle.
\end{aligned} \tag{16}
$$

Exercise 4.38. Since $\imath$ is an isomorphism between $\mathcal{H}_\mu$ and $\mathcal{R}_\mu$, a closure of $\mathcal{B}^*$ in $L^2(\mathcal{B},\mu)$, for any $h \in \mathcal{H}_\mu$,

$$
\begin{aligned}
\|h\|_\mu &= \|\imath(h)\|_{\mathcal{R}_\mu} \\
&= \sup_{l \in \mathcal{R}_\mu} \{\langle l, \imath(h)\rangle_{L^2(\mathcal{B},\mu)} : \|l\|_{L^2(\mathcal{B},\mu)} \leq 1\} \\
&= \sup_{l \in \mathcal{B}^*} \{C_\mu(l, \imath(h)) : C_\mu(l,l) \leq 1\} \\
&= \sup_{l \in \mathcal{B}^*} \{l(h) : C_\mu(l,l) \leq 1\}.
\end{aligned} \tag{17}
$$

Since the norm has to be finite, the last quantity is also finite. On the other hand, if the last quantity is finite for some $h \in \mathcal{B}$ then h determines a bounded linear functional on $\mathcal{R}_\mu$. Since $\mathcal{R}_\mu$ is a Hilbert space, by Riesz representation theorem, there is $h^* \in \mathcal{R}_\mu$ such that $l(h) = \langle l, h^*\rangle_{\mathcal{R}_\mu}$ for any $l \in \mathcal{R}_\mu$. Therefore

$$
\begin{aligned}
\sup_{l \in \mathcal{B}^*} \{l(h) : C_\mu(l,l) \leq 1\} &= \sup_{l \in \mathcal{R}_\mu} \{\langle l, h^*\rangle_{\mathcal{R}_\mu} : \|l\|_{\mathcal{R}_\mu} \leq 1\} \\
&= \|h^*\|_{\mathcal{R}_\mu} \\
&= \|h\|_\mu.
\end{aligned} \tag{18}
$$

Since the first quantity is finite, the norm is finite as well. Therefore $h \in \mathcal{H}_\mu$.