Exercise sheet 3

Exercise 4.7. Define $X = (a_0\xi_0, a_1\xi_1, \ldots)$ and suppose $\|a\|_2 < \infty$. We need to prove that X defines a measure on ℓ^2 and that this measure is Gaussian. The first claim follows from $\mathbb{E}\,\|X\|_2^2 < \infty$ which we prove below.

Let $X_n = (a_0\xi_0, \ldots, a_n\xi_n, 0, \ldots)$. It defines a Cauchy sequence in $L^2(\Omega, \ell^2)$, where Ω is a common probability space for $(\xi_0, \xi_1, \ldots)$:

$$\mathbb{E}\,\|X_n - X_m\|_2^2 = \sum_{i=m}^{n} a_i^2, \tag{1}$$

which converges to zero as $N \to \infty$ for $n > m > N$. Since $L^2(\Omega, \ell^2)$ is complete, the limit, given by X, lies in ℓ^2, i.e. $\mathbb{E}\,\|X\|_2^2 < \infty$.

This implies that $\|X\|_2$ is almost surely finite, therefore X defines a measure on ℓ^2. By definition, this measure is Gaussian iff $\langle X, b\rangle$ is Gaussian for any $b \in \ell^2$. Since all $\{\xi_n\}$ are iid centred Gaussians, this is equivalent to saying that the variance $\sum_{n=0}^{\infty}(a_n b_n)^2$ is finite. This quantity is finite for every $b \in \ell^2$ iff $a \in \ell^\infty$. The latter condition is satisfied since $a \in \ell^2 \subset \ell^\infty$.

Suppose now $\|a\|_2 = \infty$. We claim that $\|X\|_2 = \infty$ almost surely in this case, therefore X does not even define a measure on ℓ^2. W.l.o.g. assume that $\|a\|_\infty < \infty$; otherwise, X, even if it defined a measure on ℓ^2, could not be Gaussian. One has

$$\mathbb{E}\left[\left(\|X_n\|_2^2 - \mathbb{E}\,\|X_n\|_2^2\right)^2\right] = \mathbb{E}\left[\left(\sum_{i=1}^{n} a_i^2(\xi_i^2 - 1)\right)^2\right] = \mathbb{E}\left[\sum_{i=1}^{n} a_i^4(\xi_i^2 - 1)^2\right] = 2\sum_{i=1}^{n} a_i^4 \leq 2\|a\|_\infty^2 \sum_{i=1}^{n} a_i^2. \tag{2}$$

Therefore by Markov's inequality,

$$\mathcal{P}\left[\|X_n\|_2^2 \geq \frac{1}{2}\sum_{i=1}^{n} a_i^2\right] = \mathcal{P}\left[\left(\|X_n\|_2^2 - \mathbb{E}\,\|X_n\|_2^2\right)^2 < \frac{1}{4}\left(\sum_{i=1}^{n} a_i^2\right)^2\right] \geq 1 - \frac{\mathbb{E}\left[\left(\|X_n\|_2^2 - \mathbb{E}\,\|X_n\|_2^2\right)^2\right]}{\frac{1}{4}\left(\sum_{i=1}^{n} a_i^2\right)^2} \geq 1 - \frac{8\|a\|_\infty^2}{\sum_{i=1}^{n} a_i^2}. \tag{3}$$

Taking limsup as $n \to \infty$, we get $\mathcal{P}[\|X\|_2^2 \geq \|a\|_2^2] \geq 1 - \frac{8\|a\|_\infty^2}{\|a\|_2^2}$, therefore $\|X\|_2 = \infty$ almost surely whenever $\|a\|_2 = \infty$.

Exercise 4.18. Consider a sequence $\{\ell_n\}_n$ from the unit ball of $\mathcal{B}^*$. We claim that it admits a subsequence converging pointwise to some ℓ from the same unit ball.

Indeed, as long as $\mathcal{B}$ is separable, take its countable dense subset $\{x_m\}_m$. Since all ℓ_n lie in a unit ball, $\{\ell_n(x_1)\}_n$ is a bounded sequence of real numbers, hence it admits a converging subsequence. Proceeding by induction over m and applying a diagonal argument, we find a subsequence $\{\ell_{n_k}\}_k$ converging pointwise on $\{x_m\}_m$. Since this subsequence has a uniformly bounded norm and $\{x_m\}_m$ is dense in $\mathcal{B}$, the limit $\lim_{k\to\infty} \ell_{n_k}(x)$ exists for any $x \in \mathcal{B}$. Define $\ell(x) = \lim_{k\to\infty} \ell_{n_k}(x)$. It is a linear functional by construction and it lies in a unit ball as all ℓ_{n_k} lie in a unit ball.

We claim that $\{\hat{C}_\mu(\ell_{n_k})\}_k$ converges to $\hat{C}_\mu(\ell)$ in $\mathcal{B}$ which proves that $\hat{C}_\mu(B_1)$ is sequentially compact for B_1 being a unit ball in $\mathcal{B}^*$. Since $\mathcal{B}^*$ is Polish, this is equivalent to compactness. We have:

$$\lim_{k\to\infty} \|\hat{C}_\mu(\ell_{n_k}) - \hat{C}_\mu(\ell)\| \leq \lim_{k\to\infty} \int_{\mathcal{B}} \|x\|\|\ell_{n_k}(x) - \ell(x)\|\,\mu(dx) = \int_{\mathcal{B}} \|x\| \lim_{k\to\infty} \|\ell_{n_k}(x) - \ell(x)\|\,\mu(dx) = 0, \tag{4}$$

where the limit can be moved inside the integral since the integrated function is dominated by $\|x\|^2$, which is integrable by Fernique's theorem.