Exercise sheet 2

Exercise 3.6. We are going to construct an open (hence measurable) set in $L^\infty(\mathbb{R})$ such that its pre-image under Θ is not measurable in $\mathbb{R}$.

First note that sup-distance between two non-indentical Heaviside functions is 1. Therefore an open ball $B_{1/2}(H_t)$ in $L^\infty(\mathbb{R})$ of radius $\frac{1}{2}$ centered at H_t contains no other Heaviside functions. Therefore its pre-image under Θ contains only t: $\Theta^{-1}(B_{1/2}(H_t)) = \{t\}$.

Let A be a non-measurable subset of $\mathbb{R}$. Then $\Theta(A) = \bigcup_{t\in A} B_{1/2}(H(t))$ is open in $L^\infty(\mathbb{R})$, hence also measurable.

Exercise 3.8. Recall μ and ν have densities $\mathcal{D}_\mu$ and $\mathcal{D}_\nu$ with respect to a common measure ζ. Recall the definition of total variation distance:

$$\|\mu-\nu\|_{TV} = \sup_{\|\phi\|_\infty\le 1}\left|\int \phi(x)\mu(dx) - \int \phi(x)\nu(dx)\right| = \sup_{\|\phi\|_\infty\le 1}\left|\int \phi(x)(\mathcal{D}_\mu(x)-\mathcal{D}_\nu(x))\zeta(dx)\right|. \tag{1}$$

Take $\phi(x) = 2\times 1_{D_\mu(x)>D_\nu(x)} - 1$. Then

$$\|\mu-\nu\|_{TV} \ge \left|\int |\mathcal{D}_\mu(x)-\mathcal{D}_\nu(x)|\zeta(dx)\right| = \int |\mathcal{D}_\mu(x)-\mathcal{D}_\nu(x)|\zeta(dx). \tag{2}$$

On the other hand,

$$\begin{aligned}\|\mu-\nu\|_{TV} &= \sup_{\|\phi\|_\infty\le 1}\left|\int \phi(x)(\mathcal{D}_\mu(x)-\mathcal{D}_\nu(x))\zeta(dx)\right| \\ &\le \sup_{\|\phi\|_\infty\le 1}\left(\sup_x|\phi(x)|\int |\mathcal{D}_\mu(x)-\mathcal{D}_\nu(x)|\zeta(dx)\right) \\ &= \int |\mathcal{D}_\mu(x)-\mathcal{D}_\nu(x)|\zeta(dx).\end{aligned} \tag{3}$$

Therefore we have $\|\mu-\nu\|_{TV} = \int |\mathcal{D}_\mu(x)-\mathcal{D}_\nu(x)|\zeta(dx)$. Since the original definition does not depend on ξ, the equivalent one does neither.

Exercise 3.10. For any coupling π between μ and ν,

$$\begin{aligned}\|\mu-\nu\|_{TV} &= \sup_{\|\phi\|_\infty\le 1}\left|\int \phi(x)\mu(dx) - \int \phi(x)\nu(dx)\right| \\ &= \sup_{\|\phi\|_\infty\le 1}\left|\int (\phi(x)-\phi(y))\pi(dx,dy)\right| \\ &\le \sup_{\|\phi\|_\infty\le 1}\left(\sup_{x\ne y}|\phi(x)-\phi(y)|\right)\pi(\{x\ne y\}) \\ &= 2\pi(\{x\ne y\}).\end{aligned} \tag{4}$$

Therefore $\|\mu-\nu\|_{TV} \le 2\inf_{\pi\in C(\mu,\nu)}\pi(\{x\ne y\})$.

Take π to be $\pi(dx, dx) = (\mathcal{D}_\mu(x) \wedge \mathcal{D}_\nu(x))\, \zeta(dx)$ and $\pi(dx, dy) = (\mathcal{D}_\mu(x) - \mathcal{D}_\nu(x))_+\, (\mathcal{D}_\nu(y) - \mathcal{D}_\mu(y))_+\, \zeta(dx)\, \zeta(dy)$. Then

$$\begin{aligned}
\pi(\{x \neq y\}) &= 1 - \pi(\{x = y\}) \\
&= 1 - \int (\mathcal{D}_\mu(x) \wedge \mathcal{D}_\nu(x))\, \zeta(dx) \\
&= 1 - \frac{1}{2} \int (\mathcal{D}_\mu(x) + \mathcal{D}_\nu(x) - |\mathcal{D}_\mu(x) - \mathcal{D}_\nu(x)|)\, \zeta(dx) \\
&= \frac{1}{2} \int |\mathcal{D}_\mu(x) - \mathcal{D}_\nu(x)|\, \zeta(dx) \\
&= \frac{1}{2} \|\mu - \nu\|_{TV}.
\end{aligned} \tag{5}$$

Therefore $2 \inf_{\pi \in C(\mu,\nu)} \pi(\{x \neq y\}) \leq 2\pi(\{x \neq y\}) = \|\mu - \nu\|_{TV}$.

Exercise 3.19. Recall the definition of Wasserstein distance:

$$d(\mu, \nu) = \inf_{\pi \in C(\mu,\nu)} \int d(x, y)\, \pi(dx, dy). \tag{6}$$

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Let us first prove that a set of couplings, e.g. $C(\mu, \nu)$, is tight. Fix $\epsilon > 0$. Since singletons are tight, there is a compact $K \subset \mathcal{X}$ such that $\mu(K) > 1 - \epsilon$ and $\nu(K) > 1 - \epsilon$. Then for any coupling π,

$$\begin{aligned}
\pi(K \times K) &= 1 - \pi(\mathcal{X} \times \mathcal{X} \setminus K \times K) \\
&= 1 - \pi((\mathcal{X} \setminus K) \times \mathcal{X} \cup \mathcal{X} \times (\mathcal{X} \setminus K)) \\
&\geq 1 - \pi((\mathcal{X} \setminus K) \times \mathcal{X}) - \pi(\mathcal{X} \times (\mathcal{X} \setminus K)) \\
&= 1 - \mu(\mathcal{X} \setminus K) - \nu(\mathcal{X} \setminus K) \\
&\geq 1 - 2\epsilon.
\end{aligned} \tag{7}$$

By Prokhorov's theorem, $C(\mu, \nu)$ is precompact. Let us show that it is also closed. Indeed, suppose $\{\pi_n\}_{n=1}^\infty$ is a sequence of couplings converging weakly to some measure π on $\mathcal{X} \times \mathcal{X}$, i.e. $\lim_{n\to\infty} \int \phi(x, y)\, \pi_n(dx, dy) = \int \phi(x, y)\, \pi(dx, dy)$ for any $\phi \in C_b(\mathcal{X} \times \mathcal{X})$. Take $\phi(x, y) = \bar{\phi}(y)$ for some $\bar{\phi} \in C_b(\mathcal{X})$. Then

$$\int \bar{\phi}(y)\, \pi(dx, dy) = \lim_{n\to\infty} \int \bar{\phi}(y)\, \pi_n(dx, dy) = \int \bar{\phi}(y)\, \nu(dy). \tag{8}$$

By the same argument for the x-dimension, we get that π has to be a coupling. Therefore $C(\mu, \nu)$ is closed and hence compact.

Take now a sequence of couplings $\{\pi_n\}_{n=1}^\infty$ approximating the infinum in the definition of Wasserstein distance:

$$d(\mu, \nu) = \lim_{n\to\infty} \int d(x, y)\, \pi_n(dx, dy). \tag{9}$$

Since $C(\mu, \nu)$ is compact in weak topology, this sequence has a weakly converging subsequence $\{\pi_{n_k}\}_{k=1}^\infty$ and the limit π is also a coupling. Since the metric d is lower-semicontinuous then by Portmanteau theorem,

$$\int d(x, y)\, \pi(dx, dy) \leq \liminf_{k\to\infty} \int d(x, y)\, \pi_{n_k}(dx, dy) = \lim_{n\to\infty} \int d(x, y)\, \pi_n(dx, dy) = d(\mu, \nu). \tag{10}$$

However, $d(\mu, \nu) \leq \int d(x, y)\, \pi(dx, dy)$ since it is the infinum over all couplings. Therefore we have an equality:

$$\int d(x, y)\, \pi(dx, dy) = d(\mu, \nu). \tag{11}$$