Exercise sheet 1

Exercise 2.3. We are given the following equation:

$$\partial_t u = \Delta u - au + \xi, \tag{1}$$

where $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. Consider first the solution of the homogeneous equation $\partial_t u = \Delta u - au$:

$$u(t,x) = \frac{e^{-at}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4t}} U(y)\, dy = e^{-at} e^{\Delta t}[U](x) \tag{2}$$

for some function $U : \mathbb{R} \to \mathbb{R}$. When $t \to 0+$, we get $u(0,x) = U(x)$. Therefore, in order to satisfy initial conditions, we should take $U(x) = u_0(x)$.

When ξ is present, we get by variation of constants,

$$u(t,x) = e^{-at} e^{\Delta t}[u_0](x) + \int_0^t e^{-a(t-s)} e^{\Delta(t-s)}[\xi(s,\cdot)](x)\, ds. \tag{3}$$

Define the covariance as $C(s,t,x,y) = \mathbb{E}\, u(s,x)u(t,y)$. By translation invariance, $C(s,t,x,y) = C(s,t,0,y-x)$. Suppose $u_0 \equiv 0$. In this case,

$$\begin{aligned}
C(s,t,0,x) &= \frac{1}{4\pi}\mathbb{E} \int_0^t \int_0^s \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-a(t-r)-a(s-r')}}{\sqrt{|t-r||s-r'|}} e^{-\frac{(x-y)^2}{4(t-r)} - \frac{y',2}{4(s-r')}} \xi(r,y)\xi(r',y')\, dy\, dy'\, dr\, dr' \\
&= \frac{1}{4\pi}\mathbb{E} \int_0^{s\wedge t} \int_{\mathbb{R}} \frac{e^{-a(s+t-2r)}}{\sqrt{|t-r||s-r|}} e^{-\frac{(x-y)^2}{4(t-r)} - \frac{y^2}{4(s-r)}}\, dy\, dr \\
&= \frac{1}{2} \int_0^{s\wedge t} \frac{e^{-a(s+t-2r)}}{\sqrt{s+t-2r}} e^{-\frac{x^2}{4(s+t-2r)}}\, dr \\
&= \frac{1}{4} \int_{|s-t|}^{s+t} \frac{e^{-al}}{\sqrt{l}} e^{-\frac{x^2}{4l}}\, dl.
\end{aligned} \tag{4}$$

Define $G(x) = \lim_{t\to\infty} C(t,t,0,x)$. Then

$$G(x) = \int_0^\infty e^{-\frac{x^2}{4r} - ar} \frac{dr}{4\sqrt{r}}. \tag{5}$$

We have

$$\begin{aligned}
G'(x) &= \int_0^\infty -\frac{x}{2r} e^{-\frac{x^2}{4r} - ar} \frac{dr}{4\sqrt{r}} \\
&= \frac{1}{2} \int_0^\infty e^{-\frac{x^2}{4r} - ar}\, d\left(\frac{x}{2\sqrt{r}}\right) \\
&= -\frac{1}{2} \int_0^\infty e^{-z^2 - \frac{ax^2}{4z^2}}\, dz \\
&= -\sqrt{a} \int_0^\infty e^{-al - \frac{x^2}{4l}} \frac{dl}{4\sqrt{l}} \\
&= -\sqrt{a} G(x).
\end{aligned} \tag{6}$$

Therefore $G(x) = G(0)e^{-\sqrt{a}x}$, where $G(0) = \int_0^\infty \frac{e^{-ar}}{4\sqrt{r}}\, dr$.

Exercise 2.4 Recall how S^α_λ acts on functions and distributions:

$$(S^\alpha_\lambda f)(t,x) = \lambda^{-\alpha} f(\lambda^2 t, \lambda x), \qquad (S^\alpha_\lambda \xi)(\phi) = \xi(S^{-d-2-\alpha}_{1/\lambda}\phi). \tag{7}$$

Recall the canonical embedding of a function $f:\mathbb{R}^n \to \mathbb{R}$ is a distribution acting as $f(\phi) = \int_{\mathbb{R}^n} f(x)\phi(x)\,dx$. What we get for the canonical embedding of a function $f:\mathbb{R}^{d+1}\to\mathbb{R}$ is therefore

$$\begin{aligned}
f(S^{-d-2-\alpha}_{1/\lambda}\phi) &= \int_{\mathbb{R}}\int_{\mathbb{R}^d} f(t,x)(S^{-d-2-\alpha}_{1/\lambda}\phi)(t,x)\,dx\,dt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^d} f(t,x)\lambda^{-d-2-\alpha}\phi(t/\lambda^2, x/\lambda)\,dx\,dt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^d} f(\lambda^2 s, \lambda y)\lambda^{-\alpha}\phi(s,y)\,dy\,ds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^d} (S^\alpha_\lambda f)(s,y)\phi(s,y)\,dy\,ds \\
&= (S^\alpha_\lambda f)(\phi).
\end{aligned} \tag{8}$$

As for T_h defined below, we perform similar calculations:

$$(T_h f)(t,x) = f(t-h_t, x-h_x), \qquad (T_h\xi)(\phi) = \xi(T_{-h}\phi). \tag{9}$$

$$\begin{aligned}
f(T_{-h}\phi) &= \int_{\mathbb{R}}\int_{\mathbb{R}^d} f(t,x)(T_{-h}\phi)(t,x)\,dx\,dt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^d} f(t,x)\phi(t+h_t, x+h_x)\,dx\,dt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^d} f(s-h_t, y-h_x)\phi(s,y)\,dy\,ds \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^d} (T_h f)(s,y)\phi(s,y)\,dy\,ds \\
&= (T_h f)(\phi).
\end{aligned} \tag{10}$$

Let ξ be space-time white noise. Its correlation function is given by $\mathbb{E}\,\xi(\phi)\xi(\psi) = \int_{\mathbb{R}}\int_{\mathbb{R}^d}\phi(t,x)\psi(t,x)\,dx\,dt$. Let us check that $S^{-\frac{d+2}{2}}_\lambda\xi$ has the same correlation function:

$$\begin{aligned}
\mathbb{E}\,(S^{-\frac{d+2}{2}}_\lambda\xi)(\phi)(S^{-\frac{d+2}{2}}_\lambda\xi)(\psi) &= \mathbb{E}\,\xi(S^{-\frac{d+2}{2}}_{1/\lambda}\phi)\xi(S^{-\frac{d+2}{2}}_{1/\lambda}\psi) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^d}(S^{-\frac{d+2}{2}}_{1/\lambda}\phi)(t,x)(S^{-\frac{d+2}{2}}_{1/\lambda}\psi)(t,x)\,dx\,dt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^d}\lambda^{-d-2}\phi(t/\lambda^2, x/\lambda)\psi(t/\lambda^2, x/\lambda)\,dx\,dt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^d}\phi(s,y)\psi(s,y)\,dy\,ds.
\end{aligned} \tag{11}$$

Since both $\xi(\phi)$ and $(S^{-\frac{d+2}{2}}_\lambda\xi)(\phi)$ are centered Gaussians for any ϕ, coincidence of their correlation functions implies that ξ and $S^{-\frac{d+2}{2}}_\lambda\xi$ have the same law. As for T_h, one might check $\mathbb{E}\,(T_h\xi)(\phi)(T_h\xi)(\psi) = \mathbb{E}\,\xi(\phi)\xi(\psi)$ with a straightforward calculation.

Suppose now u, $T_h u$, and $S^\beta_\lambda u$ have the same law.

$$\begin{aligned}
S^{\beta-2}_\lambda(\partial_t u)(\phi) &= (\partial_t u)(S^{-d-\beta}_{1/\lambda}\phi) \\
&= (\partial_t u)(\lambda^{-d-\beta}\phi(\cdot/\lambda^2, \cdot/\lambda)) \\
&= -u(\lambda^{-d-2-\beta}\partial_t\phi(\cdot/\lambda^2, \cdot/\lambda)) \\
&= -u(S^{-d-2-\beta}_{1/\lambda}\partial_t\phi) \\
&= (S^\beta_\lambda u)(-\partial_t\phi) = u(-\partial_t\phi) = (\partial_t u)(\phi).
\end{aligned} \tag{12}$$

Similarly, $S_\lambda^{\beta-2}(\Delta u)(\phi) = (S_\lambda^\beta u)(\Delta\phi) = (\Delta u)(\phi)$. Therefore $(\partial_t - \Delta)u$ has the same law as $S_\lambda^{\beta-2}((\partial_t - \Delta)u)$. Similar calculations give $T_h(\partial_t u)(\phi) = (\partial_t u)(\phi)$ and $T_h(\Delta u)(\phi) = (\Delta u)(\phi)$ meaning that $(\partial_t - \Delta)u$ has the same law as $T_h((\partial_t - \Delta)u)$.

Exercise 2.5 Let $\alpha \in (0,1)$, $f \in C(\mathbb{R})$, and define $(S_\lambda^\alpha f)(x) = \lambda^{-\alpha} f(\lambda x)$, $(T_h f)(x) = f(x-h)$.

If f is α-Hölder continuous then for any $h \in \mathbb{R}$, any $\lambda > 0$, and any ϕ as described in the problem,

$$\begin{aligned}
|\langle S_\lambda^\alpha T_h f, \phi\rangle| &= \left| \int_{\mathbb{R}} \lambda^{-\alpha} f(\lambda(x-h))\phi(x)\,dx \right| \\
&= \left| \int_{\mathbb{R}} \lambda^{-\alpha} [f(-\lambda h) + (f(\lambda(x-h)) - f(-\lambda h))]\phi(x)\,dx \right| \\
&\leq \left| \int_{\mathbb{R}} \lambda^{-\alpha} f(-\lambda h)\phi(x)\,dx \right| + \int_{\mathbb{R}} \lambda^{-\alpha} |(f(\lambda(x-h)) - f(-\lambda h))||\phi(x)|\,dx \\
&\leq \int_{-1}^{1} \lambda^{-\alpha}|\lambda x|^\alpha\,dx = \frac{2}{1+\alpha},
\end{aligned} \tag{13}$$

which depends neither on h, nor on λ or ϕ.

Let us now prove the opposite implication. Define

$$f_n(z) = 2^{n-1} \int_{z-2^{-n}}^{z+2^{-n}} f(x)\,dx = \langle f, \phi_{n,z}\rangle, \tag{14}$$

where $\phi_{n,z} = 2^{n-1} I_{[z-2^{-n}, z+2^{-n}]}$.

Take $z, y \in \mathbb{R}$ such that $0 < |z-y| \leq 1$ and let $\lambda = |z-y| \in (0,1]$, $h = -y/\lambda$ and $x = (z-y)/\lambda = \operatorname{sgn}(z-y)$. Then $z = \lambda(x-h)$ and $y = -\lambda h$. Therefore

$$\begin{aligned}
|f_n(z) - f_n(y)| &= |\langle f, \phi_{n,z}\rangle - \langle f, \phi_{n,y}\rangle| \\
&= |\langle f, \phi_{n,\lambda(x-h)}\rangle - \langle f, \phi_{n,-\lambda h}\rangle| \\
&= 2^{n-1}\lambda \left| \int_{x-2^{-n}/\lambda}^{x+2^{-n}/\lambda} f(\lambda(t-h))\,dt - \int_{-2^{-n}/\lambda}^{2^{-n}/\lambda} f(\lambda(t-h))\,dt \right|.
\end{aligned} \tag{15}$$

Pick $n = \lfloor -\log_2 \lambda \rfloor + 1$. Assuming w.l.o.g. $x = 1$, we arrive into

$$\begin{aligned}
|f_n(z) - f_n(y)| &\leq \left| \int_{1/2}^{3/2} f(\lambda(t-h))\,dt - \int_{-1/2}^{1/2} f(\lambda(t-h))\,dt \right| \\
&= \lambda^\alpha |\langle S_\lambda^\alpha T_h f, \phi\rangle| \leq C\lambda^\alpha.
\end{aligned} \tag{16}$$

Take now $m \geq n$:

$$
\begin{aligned}
|f_m(z) - f_m(y)| &= 2^{m-1}\lambda \left| \int_{x-2^{-m}/\lambda}^{x+2^{-m}/\lambda} f(\lambda(t-h))\,dt - \int_{-2^{-m}/\lambda}^{2^{-m}/\lambda} f(\lambda(t-h))\,dt \right| \\
&\leq \lambda 2^{n-1} \left| \int_{x-2^{-n}/\lambda}^{x+2^{-n}/\lambda} f(\lambda(t-h))\,dt - \int_{-2^{-n}/\lambda}^{2^{-n}/\lambda} f(\lambda(t-h))\,dt \right| \\
&+ \lambda \sum_{k=n}^{m-1} \left| 2^k \int_{x-2^{-k-1}/\lambda}^{x+2^{-k-1}/\lambda} f(\lambda(t-h))\,dt - 2^{k-1} \int_{x-2^{-k}/\lambda}^{x+2^{-k}/\lambda} f(\lambda(t-h))\,dt \right| \\
&+ \lambda \sum_{k=n}^{m-1} \left| 2^k \int_{-2^{-k-1}/\lambda}^{2^{-k-1}/\lambda} f(\lambda(t-h))\,dt - 2^{k-1} \int_{-2^{-k}/\lambda}^{2^{-k}/\lambda} f(\lambda(t-h))\,dt \right| \\
&\leq |f_n(z) - f_n(y)| \\
&+ \lambda \sum_{k=n}^{m-1} \left| 2^n \int_{2^{k-n}x-2^{-n-1}/\lambda}^{2^{k-n}x+2^{-n-1}/\lambda} f(\lambda(2^{n-k}s-h))\,ds - 2^{n-1} \int_{2^{k-n}x-2^{-n}/\lambda}^{2^{k-n}x+2^{-n}/\lambda} f(\lambda(2^{n-k}s-h))\,ds \right| \\
&+ \lambda \sum_{k=n}^{m-1} \left| 2^n \int_{-2^{-n-1}/\lambda}^{2^{-n-1}/\lambda} f(\lambda(2^{n-k}s-h))\,ds - 2^{n-1} \int_{-2^{-n}/\lambda}^{2^{-n}/\lambda} f(\lambda(2^{n-k}s-h))\,ds \right| \\
&\leq C\lambda^\alpha + \lambda 2^n \sum_{k=n}^{m-1} C\lambda^\alpha 2^{\alpha(n-k)} \leq \frac{2C}{1-2^{-\alpha}}\lambda^\alpha.
\end{aligned}
\tag{17}
$$

Therefore $|f(z) - f(y)| = \limsup_{m\to\infty} |f_m(z) - f_m(y)| \leq \frac{2C}{1-2^{-\alpha}}\lambda^\alpha = K|z-y|^\alpha$ which is α-Hölder continuity.