Exercise sheet 12

Exercise 7.6. By Definition 7.4, a locally mild solution (x,τ) of (7.1) is given as follows:

$$x(t) = S(t)x_0 + \int_0^t S(t-s)F(x(s))\,ds + \int_0^t S(t-s)Q\,dW(s), \tag{1}$$

which should hold almost surely for $t \le \tau$.

Suppose $\tilde{L} = L - cI$, where I is the identity operator, and $\tilde{F}(x) = F(x) + cx$ for any $x \in \mathcal{D}(F)$, for some $c \in \mathbb{R}$. Then $\mathcal{D}(\tilde{L}) = \mathcal{D}(L)$ and $\tilde{L}$ generates a semi-group $\tilde{S}$ acting as $\tilde{S}(t)x = e^{-ct}S(t)x$. The corresponding mild solution is therefore given by

$$\begin{aligned}\tilde{x}(t) &= \tilde{S}(t)x_0 + \int_0^t \tilde{S}(t-s)\tilde{F}(x(s))\,ds + \int_0^t \tilde{S}(t-s)Q\,dW(s)\\ &= e^{-ct}S(t)x_0 + \int_0^t e^{-c(t-s)}S(t-s)(F(x(s))+cx(s))\,ds + \int_0^t e^{-c(t-s)}S(t-s)Q\,dW(s)\end{aligned} \tag{2}$$

almost surely for $t \le \tau$. Let us elaborate each term separately:

$$\begin{aligned}\int_0^t e^{-c(t-s)}S(t-s)F(x(s))\,ds &= \int_0^t e^{-c(t-s)}d\left(\int_0^s S(t-u)F(x(u))\,du\right)\\ &= \int_0^t S(t-u)F(x(u))\,du - c\int_0^t e^{-c(t-s)}\int_0^s S(t-u)F(x(u))\,du\,ds;\end{aligned} \tag{3}$$

$$\begin{aligned}\int_0^t e^{-c(t-s)}\int_0^s S(t-u)F(x(u))\,du\,ds &= \int_0^t e^{-c(t-s)}S(t-s)\int_0^s S(s-u)F(x(u))\,du\,ds\\ &= \int_0^t e^{-c(t-s)}S(t-s)\left(x(s) - S(s)x_0 - \int_0^s S(s-u)Q\,dW(u)\right)\\ &= \int_0^t e^{-c(t-s)}\left(S(t-s)x(s) - S(t)x_0 - \int_0^s S(t-u)Q\,dW(u)\right);\end{aligned} \tag{4}$$

$$\begin{aligned}\int_0^t e^{-c(t-s)}S(t-s)Q\,dW(s) &= \int_0^t e^{-c(t-s)}d\left(\int_0^s S(t-u)Q\,dW(u)\right)\\ &= \int_0^t S(t-u)Q\,dW(u) - c\int_0^t e^{-c(t-s)}\int_0^s S(t-u)Q\,dW(u)\,ds.\end{aligned} \tag{5}$$

It is then easy to see that summing up, we will get $x(t)$.

Exercise 7.9. We are going to show that the heat semigroup is analytic. We propose the following analytic extension:

$$(S(te^{i\theta})f)(\xi) = \frac{1}{\sqrt{4\pi t e^{i\theta}}}\int_{\mathbb{T}^n} e^{-\frac{\|\xi-\eta\|^2}{4te^{i\theta}}}f(\eta)\,d\eta, \tag{6}$$

where $f \in C(\mathbb{T}^n, \mathbb{R}^d)$.

We start with bounding the norm of its action:

$$\|S(te^{i\theta})f\|_{L^2}^2 = \frac{1}{4\pi t}\left|\iiint e^{-\frac{\|\xi-\eta\|^2}{4te^{i\theta}}}e^{-\frac{\|\xi-\eta'\|^2}{4te^{i\theta}}}\langle f(\eta), f(\eta')\rangle\,d\eta\,d\eta'\,d\xi\right|; \tag{7}$$

$$\int e^{-\frac{\|\xi-\eta\|^2}{4te^{i\theta}}} e^{-\frac{\|\xi-\eta'\|^2}{4te^{i\theta}}}\, d\xi = \int e^{-\frac{2\|\xi-(\eta+\eta')/2\|^2+\|\eta-\eta'\|^2/2}{4te^{i\theta}}}\, d\xi = \sqrt{2\pi te^{i\theta}}e^{-\frac{\|\eta-\eta'\|^2}{8te^{i\theta}}}; \tag{8}$$

$$\|S(te^{i\theta})f\|^2_{L^2} = \frac{1}{\sqrt{8\pi t}}\left|\iint e^{-\frac{\|\eta-\eta'\|^2}{8te^{i\theta}}}\langle f(\eta), f(\eta')\rangle\, d\eta\, d\eta'\right| = \frac{1}{\sqrt{8\pi t}}\left|\iint e^{-\frac{\|r\|^2}{8te^{i\theta}}}\langle f(\eta), f(\eta+r)\rangle\, d\eta\, dr\right|. \tag{9}$$

By Cauchy-Schwarz,

$$\left|\int \langle f(\eta), f(\eta+r)\rangle\, d\eta\right| \le \|f\|^2_{L^2}. \tag{10}$$

Therefore

$$\|S(te^{i\theta})f\|^2_{L^2} \le \frac{\|f\|^2_{L^2}}{\sqrt{8\pi t}}\left|\iint e^{-\frac{r^2}{8te^{i\theta}}}\, dr\right| = \|f\|^2_{L^2}, \tag{11}$$

which gives the bound $\|S(te^{i\theta})\| \le 1$ for any $t \ge 0$ and $\theta \in [0, 2\pi]$.

Take $f \in C^1_c(\mathbb{T}^n, \mathbb{R}^d)$ and let us check that $t \to S(te^{i\theta})f$ is continuous at zero. Let $L_f < \infty$ be the Lipschitz contsant of f. We have:

$$\begin{aligned}
\|S(te^{i\theta})f - f\|^2_{L^2} &= \frac{1}{4\pi t}\left|\iiint e^{-\frac{\|\xi-\eta\|^2}{4te^{i\theta}}} e^{-\frac{\|\xi-\eta'\|^2}{4te^{i\theta}}}\langle f(\eta)-f(\xi), f(\eta')-f(\xi)\rangle\, d\eta\, d\eta'\, d\xi\right| \\
&= \frac{1}{4\pi t}\left|\iiint e^{-\frac{\|\xi-\eta\|^2}{4te^{i\theta}}} e^{-\frac{\|\xi-\eta'\|^2}{4te^{i\theta}}}\|f(\eta)-f(\xi)\|\|f(\eta')-f(\xi)\|\, d\eta\, d\eta'\, d\xi\right| \\
&\le \frac{L_f^2}{4\pi t}\mathcal{L}(\operatorname{supp} f)\sup_{\xi\in\operatorname{supp} f}\left|\iint e^{-\frac{\|\xi-\eta\|^2}{4te^{i\theta}}} e^{-\frac{\|\xi-\eta'\|^2}{4te^{i\theta}}}\|\xi-\eta\|\|\xi-\eta'\|\, d\eta\, d\eta'\right| \\
&= \frac{4t}{\pi}L_f^2\mathcal{L}(\operatorname{supp} f) \to 0,
\end{aligned} \tag{12}$$

where $\mathcal{L}(A)$ is Lebesgue measure of a measurable set A. Since $C^1_c(\mathbb{T}^n, \mathbb{R}^d)$ is dense in $C(\mathbb{T}^n, \mathbb{R}^d)$, Exercise 5.2 implies that $t \to S(te^{i\theta})$ is strongly continuous for any θ and therefore analytic.

Exercise 7.12. Take $V(u) = u^2$ and f to be an odd degree polynomial with negative leading coefficient. Let $x \ge 0$ and $|y| \le R$ for some R. Then

$$V'(x)f(x+y) \le 2x(f(x) + R\max_{t\in[-R,R]} f'(x+t)). \tag{13}$$

Since f is a polynomial, for large enough x, $\max_{t\in[-R,R]} f'(x+t)$ is dominated by a leading term, and therefore negative. Hence for these large positive x,

$$V'(x)f(x+y) \le 2xf(x) \le 0 \le V(x). \tag{14}$$

Similarly, for these large negative x,

$$V'(x)f(x+y) \le 2xf(x) \le 0 \le V(x). \tag{15}$$

Since "not too large" x lie on a compact, we can always find C (which depends on R) such that

$$V'(x)f(x+y) \le CV(x) \tag{16}$$

uniformly for these x.

Exercise 7.13.