Exercises Introduction to SPDEs

May 4, 2023

Exercise 6.19

Construct an example of a potential V such that the semigroup S from the previous example is not strongly continuous by choosing it such that $\lim_{t\to\infty} \|S(t)\| = +\infty$, even though each of the operators $S(t)$ for $t > 0$ is bounded! Hint: Choose V of the form

$$V(x) = x^3 - \sum_{n>0} \frac{1}{n^{-1}} W(\frac{x - c_n}{n^{-1}}),$$

where W is an isolated 'spike' and $\{c_n\}_{n\in\mathbb{N}}$ are suitably chosen constants.

Solution:

Recall from the lecture that it suffices to find a potential such that for each $t > 0$

$$C_t := \sup_{x\in\mathbb{R}} (V(x) - V(x+t)) \tag{0.1}$$

is finite but at the same time $C_t \to +\infty$ as $t \to 0$.
We choose V as in the hint where

$$W(x) = \begin{cases} 1 - x & \text{on } (0,1)\ , \\ x+1 & \text{on } (-1,0)\ , \\ 0 & \text{else.} \end{cases}$$

and exhibit sequences c_n such that the required properties are satisfied. Thus we find

$$V(x) - V(x+t) = -3t(x^2 + xt) - t^3 + \sum_{n>0} n\left(W(\frac{x+t-c_n}{n^{-1}}) - W(\frac{x - cn}{n^{-1}})\right)$$

- Claim: If $\liminf_{n\to\infty} \frac{c_n}{\sqrt{n}} = +\infty$ and $\liminf_{n\to\infty}(c_n - c_{n-1}) > 3$, then $C_t < \infty$ for every $t \in (0,1)$.

Indeed, note that for each n and $t \in (0,1)$ the function $x \mapsto n\big(W(\frac{x+t-c_n}{n^{-1}}) - W(\frac{x-c_n}{n^{-1}})\big)$ is supported on $(c_n - 2, c_n + 1)$ where it is bounded by $2n$. Thus for x large enough and in the support of $x \mapsto W(\frac{x+t-c_n}{n^{-1}}) - W(\frac{x-c_n}{n^{-1}})$ for some n, we find that

$$V(x) - V(x+t) \lesssim -tx^2 + 2n \lesssim -t(c_n - 2)^2 + 2n < +\infty$$

uniformly in n.

- Next, let us evaluate for $t = \frac{1}{n}$ the difference at the point $x_t = c_n - \frac{1}{n}$. We find for n large enough
$$V(x_t) - V(x_t + t) \geq -6tx_t^2 - t^3 + n \geq -6\frac{1}{n}c_n^2 + n - C$$
which converges to $+\infty$ as $n \to +\infty$ as soon as c_n^2/n grows slower than n

Thus we conclude that one can choose $c_n = n^\alpha$ for $\alpha \in (1/2, 1)$.

Exercise 6.20

Show that $\mathscr{P}_t$ maps the space $\mathscr{C}_b(\mathscr{B})$ of continuous bounded functions from $\mathscr{B}$ to $\mathbb{R}$ into itself, i.e. it has the Feller property.

Solution:

We recall that $\mathscr{P}_t\phi(x) = E\left[\phi(U_x(t))\right]$, where $U_x(t) = S(t)x + \int_0^t S(t-s)QdW(s)$.
Note that for a sequence $x_n \in \mathscr{B}$ such that $x_n \to x$ for $n \to \infty$ we have that for every $t > 0$ almost surely $U_{x_n}(t) \to U_x(t)$. Since ϕ is continuous and bounded, it follows by dominated convergence that
$$\mathscr{P}_t\phi(x_n) \to \mathscr{P}_t\phi(x)\ .$$

Exercise 6.24

Show that if $dx = Lx\,dt + QdW(t)$ has an invariant measure $\mu_\infty = \mathscr{N}\left(0, \int_0^\infty S(s)QQ^*S^*(s)ds\right)$ but there exists $x \in \mathscr{H}$ such that $\limsup_{t\to\infty} \|S(t)x\| > 0$, then one cannot have $\mathscr{P}_t^*\delta_x \to \mu_\infty$ weakly. In this sense, the statement of Proposition 6.23 is sharp.

Solution:

Let μ_t be the law of $\int_0^\infty S(t)QdW_t$. From the lecture we know that $\mu_t \to \mu_\infty$ weakly. Taking fourier transforms, we find
$$\widehat{\mathscr{P}_t^*\delta_x} = \widehat{\delta_{S(t)x}} \cdot \widehat{\mu_t}\ .$$
Thus $\mathscr{P}_t^*\delta_x \to \mu_\infty$ would imply that $\widehat{\delta_{S(t)x}} \to 1$ pointwise (since the fourier transform of Gaussian measures is non zero). Finally this would imply $S(t)x \to 0$.

Exercise 6.25

Let $\mathscr{B}$ be the space of continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ with $\lim_{|x|\to\infty} \phi(x) = 0$ and $\|\phi\| = \sup_x |\phi(x)|$. Let S be the semigroup on $\mathscr{B}$ given $(S(t)\phi)(x) = \phi(e^t x)$. Show that even though one has $\|S(t)\phi\| = \|\phi\|$ for every $t > 0$ and $\phi \in \mathscr{B}$, the only probability measure μ on $\mathscr{B}$ such that $S(t)_* \mu = \mu$ for all $t > 0$ is given by $\mu = \delta_0$.

Solution:

Let $\phi \in \mathscr{B}$, then

$$\begin{aligned} \int \phi d\mu &= \lim_{t\to\infty} \int \phi dS(t)_* \mu \\ &= \lim_{t\to\infty} \int S(t)\phi d\mu \\ &= \lim_{t\to\infty} \int \phi(e^t x) d\mu(x) \\ &= \int \mathbb{1}_0(x)\phi(0) d\mu(x) \end{aligned}$$

where we used dominated convergence in the last step. Thus μ is a multiple of δ_0. Since it is a probability measure, the claim follows.