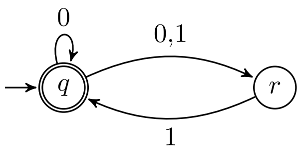
Exercise Sheet n°1

Exercise 1:

1. For each of the following languages draw the graph of a nondeterministic finite automaton (NFA) which recognises it.
   (a) $\{w \in \{0,1\}^{<\omega} \mid \text{the length of } w \text{ is a multiple of } 3\}$;
   (b) $\{w \in \{0,1\}^{<\omega} \mid w \text{ contains exactly one } 1\}$;
   (c) $\{w \in \{0,1\}^{<\omega} \mid w \text{ has length at least two and its second last letter is a } 1\}$.
2. For each of the previous languages draw the graph of a deterministic finite automaton (DFA) which recognises it.
3. Prove that any language recognised by a NFA is recognised by a DFA.

   Hint: Starting with an arbitrary NFA N, *define a* DFA D *which recognises the same language as* N *and whose set of states is the power set of the set of states of* N.
4. Convert the following NFA into a DFA recognising the same language.



Exercise 2:

Prove that on any non empty finite alphabet Σ the set $\mathcal{L}(\Sigma)$ of languages recognised by a NFA satisfies the following:

1. $\mathcal{L}(\Sigma)$ contains all finite languages,
2. $\mathcal{L}(\Sigma)$ is closed under the following operation:
   (a) complementation;
   (b) union;
   (c) concatenation of languages:
   $$LK = \{uv \in \Sigma^{<\omega} \mid u \in L \text{ and } v \in K\}, \quad \text{for } L,K \subseteq \Sigma^{<\omega};$$
   (d) the star operation,
   $$L^* = \{w_1 w_2 \cdots w_k \mid k \in \omega \text{ and each } w_i \in L\} \quad \text{for } L \subseteq \Sigma^{<\omega}.$$
3. $\mathcal{L}(\Sigma)$ is infinite countable.

Remarks. *1) In fact the Kleene Theorem states that the set of languages on a finite alphabet Σ recognised by a* DFA *is exactly the closure of the languages*

$$\emptyset,\ \{\epsilon\},\ \textit{and}\ \{a\}\ \textit{for each}\ a \in \Sigma,$$

under union, product and the star operation[1].

2) The languages recognised by a DFA *also admit an algebraic characterisation. A monoid is a set equipped with a binary associative operation and a distinguished neutral element. The set of all finite words on a finite alphabet Σ equipped with the concatenation and the empty word is denoted Σ^* and is in fact the free monoid on Σ. A language $L \subseteq \Sigma^*$ is recognisable by a* DFA *if and only if there exists a monoid morphism $\varphi : \Sigma^* \to M$ onto a finite monoid M such that for some $P \subseteq M$ we have*

$$w \in L \quad \textit{if and only if} \quad \varphi(w) \in P.$$

Exercise 3:

1. Prove the following lemma.

   Lemma (Pumping lemma)**.** *Let L be a language on a finite alphabet Σ recognised by some DFA. There exists a natural number p such that any word $w \in L$ with $|w| \geq p$ can be split into three pieces, $w = xyz$, satisfying the following properties:*

   (a) for all natural number n, $xy^nz \in L$;

   (b) $|y| > 0$;

   (c) $|xy| \leq p$.

   Hint: Consider p as the number of states of a DFA recognising L.

2. Using the Pumping lemma, show that the following languages are not recognisable by a DFA:

   (a) $\{0^n1^n \mid n \in \omega\}$;

   (b) $\{ww \mid w \in \{0{,}1\}^{<\omega}\}$;

   (c) $\{0^n \mid n \text{ is prime}\}$.

3. A well-bracketed word is a word w on the alphabet $\{(,)\}$ such that

   (a) w contains the same number of left brackets and right brackets;

   (b) for every prefix u of w, the number of left brackets in u is greater or equal to the number of right brackets in u.

   For instance, ((()())()) is a well-bracketed word while ()))()(()) is not. Show that that the language of well-bracketed words is not recognisable by a DFA.

4. Describe informally an additional feature for a DFA in order to recognise the language of well-bracketed words.

[1] Here "ϵ" denotes the empty word.