1. Denote $B = \mathcal{O}_K(A)$ and let L be the field of fractions of B. Since K is a field, we can assume that $L \subseteq K$. Suppose $x \in K$ is integral over B, i.e., $B[x]$ is finitely generated as a B-module. Since B is finitely generated as an A-module, by the lemma in class we find that $B[x]$ is finitely generated as an A-module and, hence, x is integral over A, i.e., $x \in B$. Hence B is integrally closed in K. Since $B \subseteq L \subseteq K$, this shows in particular that $\mathcal{O}_L(B) \subseteq \mathcal{O}_K(B) \subseteq B$ and, hence, B is integrally closed.
2. The norm $N(x + y\sqrt{-5}) = x^2 + 5y^2$ is multiplicative and has integer values. We have $N(2) = 4$ and $N(1 + \sqrt{-5}) = 6$. If the ideal $(2, 1 + \sqrt{-5})$ was generated by a single element $w = a + b\sqrt{5}$, the norm $N(w)$ would have to divide both 4 and 6, so it would have to divide 2. However, it is immediate to see that $N(w) = 1$ implies that w is a unit ($w = \pm 1$) and $N(w) = 2$ is not possible since there are no integer pairs (a, b) with $a^2 + 5b^2 = 2$. Therefore the ideal $(2, 1 + \sqrt{5})$ is not principal and $\mathbb{Z}[\sqrt{-5}]$ is not a P.I.D.
3. a) The fact that B is K-bilinear follows directly from the K-linearity of the map $a \mapsto \operatorname{tr}([\times a]_{L/K})$, which itself follows from linearity of the trace and of the map $[\times a]_{L/K}$.
   b)
   c) If we express $f(X) = (X - \alpha_1) \cdots (X - \alpha_n)$, for $1 \leqslant i \leqslant n$ we can compute $f'(\alpha_i) = \prod_{j \neq i} (\alpha_i - \alpha_j)$. Therefore we have

$$\prod_{i=1}^{n} \sigma_i(f'(\alpha)) = \prod_{i=1}^{n} (f'(\alpha_i)) = \prod_{1 \leqslant i,j \leqslant n,\ i \neq j} (\alpha_i - \alpha_j) = (-1)^{\frac{n(n-1)}{2}} \prod_{1 \leqslant i,j \leqslant n,\ i<j} (\alpha_i - \alpha_j)^2.$$

4. Let $\mathcal{C}$ be a $\mathbb{Z}$-basis of $\mathcal{O}_K$. Since $\mathcal{B}$ is a $\mathbb{Q}$-basis of K contained in $\mathcal{O}_K$, there exists a $d \times d$ matrix M (where d is the dimension of K over $\mathbb{Q}$) with integer coefficients and non-zero determinant such that $\mathcal{C}M = \mathcal{B}$.

   We have seen in Exercise Sheet 4 (ex. 2c) that $\Delta(\mathcal{C}) = (-4)^{r_2} \operatorname{covol}\big(f(\Lambda)\big)^2$ where Λ is the lattice generated by $\mathcal{C}$ and r_2 is the number of complex embeddings of K up to conjugation. For the same reason we have $\Delta(\mathcal{B}) = (-4)^{r_2} \operatorname{covol}\big(f(\Lambda')\big)^2$ where Λ' is the lattice generated by $\mathcal{B}$, which is a sublattice of Λ since $\mathcal{B}$ is contained in $\mathcal{O}_K$.

   Using Ex. 2 of Exercise Sheet 2, we get that $\Delta(\mathcal{B}) = \det(M)^2 \Delta(\mathcal{C})$. Therefore, if $\Delta(\mathcal{B})$ is square free, we must have $\det(M) = \pm 1$ meaning that M is invertible and thus $\mathcal{B}$ is a $\mathbb{Z}$-basis of $\mathcal{O}_K$.