1. a) We first show (ii)$\Rightarrow$(i). Suppose that a submodule $N \subseteq M$ is not of finite type. Then we can construct an infinite sequence $(x_1, x_2, x_3, \ldots)$ in N such that, for every $n \in \mathbb{N}$, we have

$$Ax_1 \subset Ax_1 + Ax_2 \subset \cdots \subset Ax_1 + Ax_2 + \cdots + Ax_n$$

where all the inclusions are strict. We construct the sequence by induction: for $n = 1$ it suffices to pick any $x_1 \in S$, then, once we have constructed the sequence up to x_n, we pick some $x_{n+1} \in N \smallsetminus (Ax_1 + Ax_2 + \cdots + Ax_n)$ which exists because otherwise we would have $N = Ax_1 + Ax_2 + \cdots + Ax_n$ and N would be of finite type. Now the sequence of submodules

$$Ax_1 \subset Ax_1 + Ax_2 \subset Ax_1 + Ax_2 + Ax_3 \subset \cdots$$

never stabilizes, contradicting (ii).
Now we show (i)$\Rightarrow$(iii). Let $\mathcal{C}$ be a collection of submodules. To show that it has a maximal element, we use Zorn's lemma. We need to show that every chain c (i. e. every subset of $\mathcal{C}$ that is totally ordered by inclusion) has an upper bound in $\mathcal{C}$. Fix a chain c and let $N := \bigcup_{L \in c} L$. By (i) N is finitely generated, so we have $N = Ay_1 + \cdots + Ay_d$ for some $y_1, \ldots, y_d \in M$. By definition of N, for $1 \leqslant i \leqslant d$ there is $N_i \in c$ with $y_i \in N_i$. Since c is totally ordered, there is a permutation σ of $\{1, \ldots, d\}$ such that $N_{\sigma(1)} \subseteq \cdots \subseteq N_{\sigma(d)}$. Therefore $y_1, \ldots, y_d \in N_{\sigma(d)}$, hence $N \subseteq N_{\sigma(d)}$. As N was the union of all the elements of c, we have $N = N_{\sigma(d)}$ and $N \in c \subseteq \mathcal{C}$ is the upper bound for the chain c.
Finally we show (iii)$\Rightarrow$(ii). Let $\mathcal{C} = (N_n)_{n \in \mathbb{N}}$ be an increasing sequence of submodules, i. e. $N_n \subseteq N_{n+1}$ for every n. By (iii) the collection $\mathcal{C}$ has a maximal element N_{n_0}. This means that $N_n \subseteq N_{n_0}$ for all n, and since the sequence is increasing this is only possible if $N_n = N_{n_0}$ for all $n \geqslant n_0$, meaning that the sequence stabilizes.

b) By hypothesis, there exist $y_1, \ldots, y_n \in M$ such that $M = Ay_1 + \cdots + Ay_n$. Therefore there is a surjective map $f : A^n \to M$, defined by $(a_1, \ldots, a_n) \mapsto \sum_{i=1}^n a_i y_i$. Every submodule $N \subset M$ is the image via f of the submodule $f^{-1}(N) \subseteq A^n$, therefore it is sufficient to show that A^n is Noetherian: once we have this, for every submodule $N \subseteq M$ the fact that $f^{-1}(N)$ is of finite type will imply that N is of finite type.
To show that A^n is Noetherian, we proceed by induction on n. The case $n = 1$ is true by hypothesis. To show that A^n Noetherian implies A^{n+1} Noetherian, we prove the following lemma: for every short exact sequence

$$0 \to L \xrightarrow{\phi} M \xrightarrow{\psi} N \to 0$$

of A-modules, if L and N are of finite type then so is M. Indeed, if $L = Ax_1 + \ldots + Ax_d$ and $N = Az_1 + \ldots + Az_l$, setting $y_i = \phi(x_i)$ for $1 \leqslant i \leqslant d$ and choosing y_{d+i} such that $\psi(y_{d+i}) = z_i$ for $1 \leqslant i \leqslant l$, we have $M = Ay_1 + \ldots + Ay_{d+l}$: for every $y \in M$ there exist $a_{d+1}, \ldots, a_{d+l} \in A$ such that $\psi(y) = \sum_{i=1}^l a_{d+i} z_i$, so $y - \sum_{i=1}^l a_{d+i} y_{d+i} \in \operatorname{Ker}(\psi) = \operatorname{Im}(\phi) = Ay_1 + \ldots + Ay_d$.

Now, to see that A^{n+1} is Noetherian, consider the short exact sequence

$$0 \to A \xrightarrow{\phi} A^{n+1} \xrightarrow{\psi} A^n \to 0$$

where $\phi(a) = (0, \ldots, 0, a)$ and $\psi(a_1, \ldots, a_n, a_{n+1}) = (a_1, \ldots, a_n)$. For every submodule $M \subseteq A^{n+1}$, the sequence

$$0 \to \phi^{-1}(A \cap M) \xrightarrow{\phi} M \xrightarrow{\psi} \psi(M) \to 0$$

is also short exact, where $A \cap M$ is the intersection of M with the submodule $\{(0, \ldots, 0, a) : a \in A\}$. Since A and A^n are Noetherian, the submodules $\phi^{-1}(A\cap M)$ and $\psi(M)$ are of finite type. The lemma implies that M is also of finite type, hence A^{n+1} is Noetherian.

2.

3.

4. Let A be a Dedekind ring and $Q = \mathrm{Frac}(A)$. Let B a domain and $Q \hookrightarrow B$ an embedding. Let $\Lambda \subseteq B$ be a non-zero A-module of finite type and $x \in B$ such that $x.\Lambda \subseteq \Lambda$.

Let $\lambda \in \Lambda \setminus \{0\}$ be an element. We have $x.\lambda \in \Lambda$ and so

$$A[x].\lambda \subseteq \Lambda.$$

Now since A is Noetherian, $A[x].\lambda$ is a finitely generated A submodule of Λ. As multiplication by λ is an A-module automorphism of $K = \mathrm{Frac}(B)$, we obtain that $A[x] \subseteq K$ is a f.g. A-submodule of K. By the equivalent characterizations of integrality, it follows that $x \in \mathcal{O}_K(A) \cap B = \mathcal{O}_B(A)$.

The second part follows since any fractional ideal $\mathfrak{f} \subseteq Q$ is a f.g. A-module and A is integrally closed.