1. Let $K/\mathbb{Q}$ a number field. Recall that $K^\times$ acts on $K_\infty = \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ by
$$\lambda\colon K^\times \times K_\infty \to K_\infty,$$
$$\big(x, (v_i)_{i=1}^r\big) \mapsto (\sigma_i(x)v_i)_{i=1}^r.$$
We let $\lambda^\times$ denote the restriction of λ to $K^\times \times K_\infty^\times$.
   a) Prove that $\lambda^\times(K^\times \times K_\infty^\times) \subseteq K_\infty^\times$, i.e., λ induces a well-defined action of $K^\times$ on $K_\infty^\times$.
   b) Prove that $K^\times$ acts freely on $K_\infty^\times$, i.e.,
$$\forall x \in K^\times \forall v \in K_\infty^\times \quad \lambda^\times(x, v) = v \implies x = 1.$$
   c) Let $H, M < K^\times$ are subgroups such that $H \cap M = \{1\}$. Prove that $HM < K^\times$ is a subgroup and $HM \cong H \oplus M$. Deduce that there is a subgroup $U < \mathcal{O}_K^\times$ such that $U \cong \mathbb{Z}^{r-1}$ and
$$\mathcal{O}_K^\times \cong \mu_K \oplus U,$$
   where $\mu_K < K^\times$ is the group of roots of unity.
   Remark: In the proof of Dirichlet's unit theorem we have proven that there is a homomorphism $F\colon \mathcal{O}_K^\times \to \mathbb{Z}^{r-1}$ such that
$$1 \longrightarrow \mu_K \longrightarrow \mathcal{O}_K^\times \xrightarrow{F} \mathbb{Z}^{r-1} \longrightarrow 0$$
   is exact. This exercise shows that the sequence splits. In class, this was stated as a consequence of the classification of finitely generated $\mathbb{Z}$-modules.
   d) Let $H, M < K^\times$ as above and suppose that H is finite. Assume that $E \subseteq K_\infty^\times$ is HM-invariant, i.e.,
$$\forall x \in HM \quad \lambda^\times(x, E) \subseteq E.$$
   Prove that there is an $|H|$-to-one correspondence between M-orbits in E and HM-orbits in E.
   e) Prove that there exists $C > 1$ such that for all $x \in K^\times$ there is $u \in \mathcal{O}_K^\times$ satisfying
$$\forall 1 \leqslant i \leqslant d \quad \frac{|\sigma_i(xu)|}{C} \leqslant \mathrm{Nr}_{K/\mathbb{Q}}(x)|^{\frac{1}{d}} \leqslant C|\sigma_i(xu)|.$$
   Hint: Let $\Lambda \subseteq \mathbb{R}^{r-1}$ a lattice. Then there is $A > 0$ depending only on Λ such that
$$\forall v \in \mathbb{R}^{r-1} \exists \ell \in \Lambda \quad \|v - \ell\| \leqslant A.$$
2. Let $K/\mathbb{Q}$ a number field of degree d and assume that $r = r_1 + r_2 \geqslant 2$. Let $(\varepsilon_1, \ldots, \varepsilon_{r-1})$ be a set of fundamental units of $\mathcal{O}_K$ and let
$$\mathcal{P} = \left\{ \sum_{i=1}^{r-1} t_i \mathrm{Log}_\infty(\varepsilon_i) \colon \forall i \in \{1, \ldots, r-1\}\, t_i \in [0, 1[\right\}.$$
Let $\mathrm{Log}\colon K_\infty^\times \to \mathbb{R}^r$ be given by
$$\mathrm{Log}(x_1, \ldots, x_r) = (d_1 \log|x_1|, \ldots, d_r \log|x_r|) \quad (x \in K_\infty^\times),$$
where $d_i = 1$ for $1 \leqslant i \leqslant r_1$ and $d_i = 2$ otherwise.

We recall that $\mathrm{Nr}\colon K_\infty^\times \to (0,\infty)$ denotes the norm-map given by

$$\forall v = (v_1,\dots,v_r) \in K_\infty^\times \quad \mathrm{Nr}(v) = \prod_{i=1}^{r} |v_i|^{d_i}.$$

Let

$$F_{\leqslant 1} = \left\{tv\colon v \in \mathrm{Log}^{-1}(\mathcal{P}), t \in]0,1]\right\}.$$

Prove that

$$\mathrm{vol}(F_{\leqslant 1}) = 2^{r_1}\pi^{r_2}\mathrm{reg}(\mathcal{O}_K).$$

Hint: Recall the outline of the argument presented in class:

a) Prove that $v \in F_{\leqslant 1}$ if and only if $\mathrm{Nr}(v) \leqslant 1$ and $\mathrm{Log}\big(\mathrm{Nr}(v)^{-\frac{1}{d}}v\big) \in \mathcal{P}$.

b) Let $\mathrm{Log}_{\mathrm{abs}}\colon (0,\infty)^r \to \mathbb{R}^r$ be the map given by

$$\forall T = (T_1,\dots,T_r) \in (0,\infty)^r \quad \mathrm{Log}_{\mathrm{abs}}(T) = (d_i \log T_i)_{i=1}^{r}.$$

Let $L\colon \mathbb{R}^r \to \mathbb{R}^r$ be given by

$$\forall v = (v_1,\dots,v_r) \in \mathbb{R}^r \quad L(v) = \sum_{i=1}^{r} v_i.$$

Show that

$$\mathrm{vol}(F_{\leqslant 1}) = 2^{r_1}(2\pi)^{r_2}\int_{\mathrm{Log}_{\mathrm{abs}}\left(]0,1]\mathrm{Log}_{\mathrm{abs}}^{-1}(\mathcal{P})\right)} \exp(Lv)\mathrm{d}v.$$

c) Let $\mathbf{1} \in \mathbb{R}^r$ be the vector with coordinates all equal to 1. Show that the following are equivalent.

(i) $v \in \mathrm{Log}_{\mathrm{abs}}\left(]0,1]\mathrm{Log}_{\mathrm{abs}}^{-1}(\mathcal{P})\right)$.

(ii) $Lv \leqslant 0$ and $v - \frac{L(v)}{d}\mathbf{1} \in \mathcal{P}$.

d) Denoting by $\mathbb{I}_A$ the indicator function on a set A, deduce that

$$\mathrm{vol}(F_{\leqslant 1}) = \int_{\mathbb{R}} \mathbb{I}_{]0,\infty]}(Lv)\mathbb{I}_{\mathcal{P}}\big(v - L(v)\mathbf{1}\big)e^{L(v)}\mathrm{d}v$$

and conclude that $\mathrm{vol}(F_{\leqslant 1}) = 2^{r_1}\pi^{r_2}\mathrm{reg}(\mathcal{O}_K)$.

3. Let K be a quadratic number field, let $D = |\mathrm{disc}(K)|$, and let χ_K be the quadratic character associated with K, i.e., χ_K is the multiplicative extension of the map

$$\chi_K(p) = \begin{cases} 1 & \text{if } p \text{ splits in } K, \\ -1 & \text{if } p \text{ is inert in } K, \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$L(s,\chi_K) = \sum_{n\in\mathbb{N}} \frac{\chi_K(n)}{n^s} \quad (\mathrm{Re}(s) > 1).$$

a) Show that for $\mathrm{Re}(s) > 1$, ζ_K admits an Euler product, i.e.,

$$\zeta_K(s) = \prod_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)\smallsetminus\{0\}} \left(1 - \frac{1}{\mathrm{Nr}(\mathfrak{p})^s}\right)^{-1}.$$

b) Prove that

$$\zeta_K(s) = \zeta(s)L(s,\chi_K).$$

4. We recall the definition of a Dirichlet character. Given $q \in \mathbb{N}$, a Dirichlet character mod q is a map $\psi\colon \mathbb{N} \to \mathbb{C}$ for which there exists a character $\chi\colon (\mathbb{Z}/q\mathbb{Z})^\times \to \mathbb{C}$ such that

$$\forall n \in \mathbb{N} \quad \psi(n) = \begin{cases} \chi(n \mod q) & \text{if } (n,q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Given a Dirichlet character ψ, we define the Dirichlet L-function

$$L(\psi, s) = \sum_{n \geqslant 1} \frac{\psi(n)}{n^s} \quad (\operatorname{Re} s > 1).$$

The trivial Dirichlet character ψ_0 mod q is the Dirichlet character induced by the unit character, i.e., $\psi_0(n) = 1$ for all $(n,q) = 1$.

a) Convince yourself (to any degree of detail you wish) that for any Dirichlet character ψ mod q, we have

$$L(\psi, s) = \prod_p \left(1 - \frac{\psi(p)}{p^s}\right)^{-1} \quad (\operatorname{Re} s > 1).$$

b) Prove that

$$L(\psi_0, s) = \zeta(s) \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right) \quad (\operatorname{Re} s > 1).$$

c) Convince yourself (to any degree of detail you wish) that for any non-trivial Dirichlet character ψ mod q, the Dirichlet L-function admits a holomorphic extension to $\{s\colon \operatorname{Re}(s) > 0\}$.
Hint: What is $\sum_{n=1}^{q} \psi(n)$?

d) Let $K = \mathbb{Q}(\zeta_q)$ and let

$$G_K(s) = \prod_{\mathfrak{p} \mid q} \left(1 - \frac{1}{\operatorname{Nr}(\mathfrak{p})^s}\right)^{-1}.$$

Show that

$$\zeta_K(s) = G_K(s) \prod_{\psi} L(\psi, s),$$

where ψ runs over all Dirichlet characters mod q.

e) Prove that for $\psi \neq \psi_0$, we have $L(\psi, 1) \neq 0$.

f) Let ψ be a Dirichlet character mod q. Show that on $\{s\colon \operatorname{Re} s > 1\}$, the Dirichlet L-function $L(\psi, s)$ admits an analytic logarithm such that

$$\log L(\psi, s) = \sum_p \frac{\chi(p)}{p^s} + g_\psi(s),$$

where g_ψ is holomorphic on $\{s\colon \operatorname{Re} s > 3/4\}$.

g) *Dirichlet's theorem on primes in arithmetic progressions.* Let $q > 1$ and suppose that $(a, q) = 1$. Prove that there are infinitely many primes $p \in \mathbb{N}$ satisfying $p \equiv a \mod (q)$.
Hint: Let $a^{-1} \in \mathbb{N}$ such that $a.a^{-1} \equiv 1 \mod (q)$. Prove that

$$\sum_{\psi} \psi(a^{-1}) \log L(\psi, s) = \sum_{p \equiv a \mod (q)} \frac{\varphi(q)}{p^s} + g(s) \quad (\operatorname{Re} s > 1),$$

where g is holomorphic on $\{s\colon \operatorname{Re} s > 3/4\}$.