PROOF OF RIESZ-THORIN

1. Riesz-Thorin

We now prove the Riesz-Thorin interpolation theorem, interestingly by using complex analysis! The key technical ingredient from complex analysis is the following

Lemma 1.1. *(Hadamard three lines lemma) Let F be a complex analytic function on the strip $S := \{z \in \mathbf{C} | \, 0 < \operatorname{Re} z < 1\}$, which extends continuously and boundedly to the closure $\bar{S}$. Assume the bounds*

$$|F(z)| \leq B_0, \ \operatorname{Re} z = 0, \ |F(z)| \leq B_1, \ \operatorname{Re} z = 1$$

for positive constants B_0, B_1. Then we have

$$|F(z)| \leq B_0^{1-\theta} B_1^{\theta}, \ \operatorname{Re} z = \theta, \forall \theta \in [0,1]$$

Proof. This is an application of the maximum modulus principle. Introduce the function

$$f(z) := F(z)[B_0^{1-z} B_1^{z}]^{-1}$$

Observe that $|[B_0^{1-z} B_1^{z}]^{-1}| \leq \max\{B_0^{-1}, B_1^{-1}\}$ provided $\operatorname{Re} z \in [0,1]$, and so $f(z)$ is also bounded on $\bar{S}$. Moreover, letting

$$f_{\varepsilon}(z) := f(z) e^{\varepsilon[z^2-1]}$$

for small $\varepsilon > 0$, we have for $z = x + iy$

$$|f_{\varepsilon}(z)| = |f(z)| e^{\varepsilon(x^2-y^2-1)}$$

which converges to zero as $|y| \to \infty$ while $x \in [0,1]$. Moreover, we have $|f_{\varepsilon}(z)| \leq 1$ if $x = 0, 1$. Now pick $y_0 > 0$ sufficiently large such that

$$|f_{\varepsilon}(z)| \leq 1, \ |y| \geq y_0.$$

Then by the maximum modulus principle and since the boundary values on the rectangular box $0 \leq x \leq 1$, $|y| \leq y_0$ are at most 1 by the preceding, we infer

$$|f_{\varepsilon}(z)| \leq \forall z \in \bar{S}.$$

Letting $\varepsilon \to 0$ we find

$$|f(z)| \leq 1 \, \forall z \in \bar{S},$$

which implies the lemma. □

Next, the proof of Riesz-Thorin:

Proof. Let $p_0 \ldots, q_1$ as in the theorem, and assume

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \ \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \ \theta \in (0,1).$$

Below, we shall assume that all $p_0 \ldots, q_1$ are strictly between 1 and ∞, leaving that exceptional case as an exercise.

Consider a simple function

$$f(x) = \sum_{k=1}^{M} a_k e^{i\alpha_k} \chi_{A_k}(x)$$

where the measurable sets $A_k \subset X$ are disjoint, and $a_k \in \mathbb{R}_{>0}$. We have

$$\|Tf\|_{L^q} = \sup_{\|g\|_{L_{q'}} \leq 1} \Big| \int_X T(f)(x) g(x) d\mu \Big|,$$

Here we may restrict g to simple functions, by density of these in $L^q(X)$. We shall now pick an arbitrary simple function

$$g(x) = \sum_{k=1}^{N} b_k e^{i\beta_k} \chi_{B_k}(x)$$

So far there has been no complex analysis involved, but now introduce(!)

$$f_z(x) := \sum_{k=1}^{M} a_k^{P(z)} e^{i\alpha_k} \chi_{A_k}(x),\ g_z(x) = \sum_{k=1}^{N} b_k^{Q(z)} e^{i\beta_k} \chi_{B_k}(x)$$

where we set

$$P(z) := \frac{p}{p_0}(1-z) + \frac{p}{p_1} z,\ Q(z) := \frac{q'}{q_0'}(1-z) + \frac{q'}{q_1'} z.$$

We shall restrict $\operatorname{Re} z \in [0,1]$, i. e. the strip S. Then we easily have that the function

$$F(z) := \int_X T(f_z)(x) g_z(x) d\mu = \sum_{k,l} a_k^{P(z)} b_l^{Q(z)} e^{i\alpha_k} e^{i\beta_l} \int_X T(\chi_{A_k})(x) \chi_{B_l}(x)\, d\mu$$

is analytic on S and continuous and bounded on $\bar{S}$ (of course the sum has only finitely many terms).

Our strategy shall be to apply the Hadamard lemma to F. For this we need good bounds on the boundary $\operatorname{Re} z = 0, 1$.

If $\operatorname{Re} z = 0$, we have $\operatorname{Re} P(z) = \frac{p}{p_0}$, and then

$$\|f_z\|_{L^{p_0}}^{p_0} = \sum_k a_k^p |A_k| = \|f\|_{L^p}^p,\ \|g_z\|_{L^{q_0'}}^{q_0'} = \sum_k b_k^{q'} |B_k| = \|g\|_{L^{q'}}^{q'}$$

By the same token, if $\operatorname{Re} z = 1$,

$$\|f_z\|_{L^{p_1}}^{p_1} = \sum_k a_k^p |A_k| = \|f\|_{L^p}^p,\ \|g_z\|_{L^{q_1'}}^{q_1'} = \sum_k b_k^{q'} |B_k| = \|g\|_{L^{q'}}^{q'}$$

Thus by the assumed bounds on p_0, we get for $\operatorname{Re} z = 0$

$$|F(z)| \le \|T(f_z)\|_{L^{q_0}} \|g_z\|_{L^{q_0'}} \le A_0 \|f\|_{L^p}^{\frac{p}{p_0}} \|g\|_{L^{q'}}^{\frac{q'}{q_0'}},$$

while if $\operatorname{Re} z = 1$, we have

$$|F(z)| \le \|T(f_z)\|_{L^{q_1}} \|g_z\|_{L^{q_1'}} \le A_1 \|f\|_{L^p}^{\frac{p}{p_1}} \|g\|_{L^{q'}}^{\frac{q'}{q_1'}},$$

Then apply the Hadamard's three line lemma with

$$B_0 = A_0 \|f\|_{L^p}^{\frac{p}{p_0}} \|g\|_{L^{q'}}^{\frac{q'}{q_0'}},\ B_1 = A_1 \|f\|_{L^p}^{\frac{p}{p_1}} \|g\|_{L^{q'}}^{\frac{q'}{q_1'}}.$$

It follows that

$$|F(z)| \le \left(A_0 \|f\|_{L^p}^{\frac{p}{p_0}} \|g\|_{L^{q'}}^{\frac{q'}{q_0'}}\right)^{1-\theta} \left(A_1 \|f\|_{L^p}^{\frac{p}{p_1}} \|g\|_{L^{q'}}^{\frac{q'}{q_1'}}\right)^{\theta}$$

provided $\operatorname{Re} z = \theta$. In particular, if $z = \theta$, we have

$$P(z) = \frac{p}{p_0}(1-\theta) + \frac{p}{p_1}\theta = 1,\ Q(z) = 1,$$

so $f_z = f, g_z = g$, and

$$|F(z)| = \left| \int_X T(f) g\, d\mu \right| \le A_0^{1-\theta} A_1^{\theta} \|f\|_{L^p} \|g\|_{L^{q'}}$$

Riesz-Thorin follows. □