DISPERSIVE PDE 23, PROBLEM SET 9

(1) Let (X, d) be a complete metric space and let $T_j, j = 1, 2$ be two contractions on X satisfying
$$d(T_j(x), T_j(y)) \leq \alpha d(x, y) \, \forall x, y \in X$$
and some $\alpha \in [0, 1)$, and further assume that
$$d(T_1(x), T_2(x)) \leq \delta$$
for some $\delta \geq 0$. Then show that the unique fixed points $p_j, j = 1, 2$ of T_j satisfy
$$d(p_1, p_2) \leq \frac{\delta}{1 - \alpha}.$$

(2) Use the preceding and the argument for constructing local solutions for NLS in lecture5.pdf to conclude the local Lipschitz dependence of the solutions on the initial data in $H^s(\mathbb{R}^n)$, $s > \frac{n}{2}$. Thus if I is the interval constructed there on which a local solution exists for data in some $B_a(0) \subset H^s(\mathbb{R}^n)$, $s > \frac{n}{2}$, then show that there is a constant C such that
$$\big\|\psi_1 - \psi_2\big\|_{L_t^\infty H^s(I \times \mathbb{R}^n)} \leq C \cdot \big\|f_1 - f_2\big\|_{H^s(\mathbb{R}^n)}$$
for all $f_{1,2} \in B_a(0)$, where ψ_j is the solution of
$$i\psi_{j,t} + \triangle\psi_j = \pm|\psi|^{p-1}\psi_j, \, \psi_j(0, \cdot) = f_j(\cdot),$$
and we assume $p \in 2\mathbb{N} + 1$.

(3) Show that the nonlinear wave equation
$$\Box\psi = |\psi|^{p-1}\psi, \, \psi[0] = (f, g),$$
is strongly locally well-posed in $H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$, provided $s > \max\{\frac{n}{2}, 1\}$. For this follow the argument for the Schrodinger case.