DISPERSIVE PDE 23, PROBLEM SET 8

(1) Check that there is a constant C only depending on our choice of the cutoff defining the P_l such that for any $p \in [1,\infty]$ we have

$$\|P_{<l}\psi\|_{L^p(\mathbb{R}^n)} \leq C\|\psi\|_{L^p(\mathbb{R}^n)},\ \|P_{\geq l}\psi\|_{L^p(\mathbb{R}^n)} \leq C\|\psi\|_{L^p(\mathbb{R}^n)}$$

(2) Check that we have the inequalities

$$C_1\|P_{[l-10,l+10]}\psi\|_{H^s(\mathbb{R}^n)} \leq 2^{ls}\cdot\|P_{[l-10,l+10]}\psi\|_{L^2(\mathbb{R}^n)} \leq C_2\|P_{[l-10,l+10]}\psi\|_{H^s(\mathbb{R}^n)}$$

provided $s \geq 0$, $l \geq 0$, and $C_{1,2}$ are suitable positive constants.

(3) Verify carefully the Littlewood-Paley decomposition

$$\begin{aligned} P_l(f\cdot g) &= P_l(P_{<l-10}f\cdot P_{[l-5,l+5]}g) \\ &+ P_l(P_{[l-10,l+10]}f\cdot P_{<l+15}g) \\ &+ \sum_{|k_1-k_2|<5,\, k_1>l+10} P_l(P_{k_1}f\cdot P_{k_2}g) \end{aligned}$$

(4) Carefully check that

$$C_1\|\psi\|^2_{H^s(\mathbb{R}^n)} \leq \sum_l \|P_l\psi\|^2_{H^s(\mathbb{R}^n)} \leq C_2\|\psi\|^2_{H^s(\mathbb{R}^n)}.$$

for suitable positive constants $C_{1,2}$.