DISPERSIVE PDE 23, PROBLEM SET 6

(1) Provide the details that if $f \in \mathcal{S}(\mathbb{R}^n)$ then we have
$$\lim_{N_{1,2}\to+\infty} \big\|P_{[-N_1,N_2]}f\big\|_{L^p(\mathbb{R}^n)} = \big\|f\big\|_{L^p(\mathbb{R}^n)},\ 1 < p \leq \infty.$$

(2) Assuming the fundamental theorem of Littlewood-Paley theory (Theorem 3. 2 in Lecture3.pdf), deduce the following weaker version of Mikhlin's theorem (Theorem 3.4 in Lecture3.pdf): if m is a Mikhlin Fourier multiplier, then assuming the fundamental theorem of LP-theory, we have
$$\big\|Tf\big\|_{L^p(\mathbb{R}^n)} \leq C \cdot \big(\sum_l \big\|P_l f\big\|_{L^p(\mathbb{R}^n)}^2\big)^{\frac{1}{2}},\ 2 \leq p < \infty.$$
This would remain correct for $p = +\infty$ if we assumed the LP-theorem was valid in that case.

(3) Use the preceding exercise to show that if $n = 1$ the Littlewood-Paley theorem does not hold for $p = \infty$. For this you may also assume that the classical Hilbert transform
$$Hf := \frac{i}{\pi} \cdot \int_{P.V} \frac{f(y)}{x-y}\,dy := \lim_{\epsilon\to 0} \frac{i}{\pi} \cdot \int_{|x-y|\geq\epsilon} \frac{f(y)}{x-y}\,dy$$
corresponds to the Fourier multiplier $m(\xi) = \frac{\xi}{|\xi|}$. For this, let f be the characteristic function of $[0,1]$, and let
$$g_\epsilon(x) = \phi_\epsilon * f$$
where $\phi_\epsilon = \epsilon^{-1}\phi(\frac{x}{\epsilon})$and ϕ is a smooth non-negative function supported in $[-1,1]$ and with $\int \phi\,dx = 1$. Then show that
$$\big\|Hg_\epsilon\big\|_{L^\infty(\mathbb{R})} \geq C|\log\epsilon|,$$
while on the other hand we have
$$\big(\sum_l \big\|P_l g_\epsilon\big\|_{L^\infty(\mathbb{R})}^2\big)^{\frac{1}{2}} \leq D|\log\epsilon|^{\frac{1}{2}}.$$
To show this last inequality, use that
$$P_l g_\epsilon(x) = (2\pi)^{-1} \int_{\mathbb{R}} \phi_\epsilon(x-y) \cdot \int \psi_l(\xi) e^{iy\xi} \cdot \frac{e^{-i\xi}-1}{\xi}\,d\xi\,dy$$
and for fixed $x \in \mathbb{R}$ distinguish between $2^l\epsilon \leq 1, 2^l\epsilon > 1$.