DISPERSIVE PDE 23, PROBLEM SET 5

(1) Let $f \in \mathcal{S}(\mathbb{R}^n)$ have Fourier support in the set $\{a2^j \leq |\xi| \leq b2^j\}$ where a, b are two positive constants. Show that there is a constant $C = C(a, b)$ and such that

$$\big\|f\big\|_{L^q(\mathbb{R}^n)} \leq C \cdot 2^{jn(\frac{1}{p}-\frac{1}{q})} \cdot \big\|f\big\|_{L^p(\mathbb{R}^n)},\ 1 \leq p \leq q \leq \infty.$$

(2) Prove Hormander's theorem: let $T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ be a bounded and translation invariant operator. This means that $\tau_h \circ T = T \circ \tau_h$ for all $h \in \mathbb{R}^n$, where $(\tau_h f)(x) = f(x+h)$. Then necessarily $q \geq p$. To show this, proceed as follows:
(i) If $f, g \in L^p(\mathbb{R}^n)$, $p \in [1, \infty)$, then $\lim_{|h|\to\infty} \|\tau_h(f)+g\|_{L^p(\mathbb{R}^n)} = (\|f\|_{L^p(\mathbb{R}^n)}^p + \|g\|_{L^p(\mathbb{R}^n)}^p)^{\frac{1}{p}}$.
(ii) Set $f = g$ for suitable $f \in L^p(\mathbb{R}^n)$ and deduce a contradiction if $q < p$.

(3) Show that if $\|\hat{f}\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$ for all $f \in \mathcal{S}(\mathbb{R}^n)$ and some fixed constant C, then necessarily $\frac{1}{q} + \frac{1}{p} = 1$.

(4) Show that if the function $f : \mathbb{R}^n \to \mathbb{R}$ is measurable, then we have $(1 \leq p < \infty)$

$$\int_{\mathbb{R}^n} |f|^p\,dx = \int_0^\infty p\lambda^{p-1} \cdot \big|\{x \in \mathbb{R}^n \,|\, |f(x)| > \lambda\}\big|\,d\lambda$$

First consider simple functions (i. e. linear combinations of characteristic functions of measurable sets)