DISPERSIVE PDE 23, PROBLEM SET 4

(1) Verify that if we set for $k \in \mathbb{N}$
$$\Big\|f\Big\|_{H^k(\mathbb{R}^n)} := \sum_{|\alpha|\leq k} \| \prod_{j=1}^n \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}} f \|_{L^2(\mathbb{R}^n)},$$
then there exist positive constants $C_{k,n}, D_{k,n}$ with the property that
$$C_{k,n}\big\|f\big\|_{H^k(\mathbb{R}^n)} \leq \Big\|f\Big\|_{H^k(\mathbb{R}^n)} \leq D_{k,n}\big\|f\big\|_{H^k(\mathbb{R}^n)},$$
where we define $\big\|\cdot\big\|_{H^k(\mathbb{R}^n)}$ as in class via the Fourier transform.

(2) Use the previous exercise to verify that
$$H^1(\mathbb{R}^2) \notin L^\infty(\mathbb{R}^2).$$
For this consider the function
$$f(x) = \chi(|x|) \cdot \log\big|\log|x|\big|,$$
where χ is a bump function smoothly localizing to $|x| \leq \frac{1}{2}$ and identically equal to 1 around 0, and $|x|$ denotes the usual length of the vector $x \in \mathbb{R}^2$.

(3) Show by means of a (more or less) explicit example that
$$H^{\frac{1}{2}}(\mathbb{R}) \notin L^\infty(\mathbb{R}).$$

(4) Define the Holder semi norm by
$$\big\|f\big\|_{C^\alpha(\mathbb{R}^n)} := \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha},\ \alpha \in (0,1).$$
Then strengthen the Sobolev embedding proved in class as follows: if $f \in H^s(\mathbb{R}^n)$, $s > \frac{n}{2}$, then provided $0 < \alpha < \min\{s - \frac{n}{2}, 1\}$ we have
$$\big\|f\big\|_{C^\alpha(\mathbb{R}^n)} \leq C_{\alpha,s,n} \cdot \big\|f\big\|_{H^s(\mathbb{R}^n)}.$$