DISPERSIVE PDE 23, PROBLEM SET 3

(1) Let $r < -1$. Construct a sequence of bump functions $\chi_l(x)$ as in the lower half of p. 8 of lecture1.pdf. For this set first

$$\tilde{\chi}_l(x) = \chi(\frac{x-2^l}{2^l}) + \chi(\frac{x+2^l}{2^l}),\ l \geq 1,$$
$$\tilde{\chi}_0(x) = \chi(x),$$

where $\chi \in C_0^\infty(\mathbb{R})$ is a suitably chosen bump function, such that

$$\sum_{l=0}^{\log_2 |r|+1} \tilde{\chi}_l(x) \geq 1$$

for any $x \in [-2|r|, 2|r|]$(show that this is possible). Then modify the $\tilde{\chi}_l(x)$ suitably to obtain the desired $\chi_l(x)$.

(2) Show that the integral at the end of lecture1.pdf on p. 9 can be bounded uniformly in $r < -1$.

(3) Give an interpretation of the expression

$$|\nabla_x u|^2 - (\omega \cdot \nabla_x u)^2$$

where $\omega = \frac{x}{|x|}$ for some fixed $x_0 \in \mathbb{R}^n$ in terms of 'angular derivatives', which are given by

$$\frac{x_i}{|x|}\partial_{x_j} u - \frac{x_j}{|x|}\partial_{x_i} u$$

where i, j run over unequal pairs of integers $1, 2 \ldots, n$.

(4) Let $u \in C^2(\mathbb{R}^{1+n})$ solve $\Box u = 0$; also assume $u(t, \cdot) \in C_0^2(\mathbb{R}^{1+n})$ for each $t \in \mathbb{R}$. Consider the 'weighted momentum functional'

$$F(t) := \int_{\mathbb{R}^n} \sum_{j=1}^n x_j u_t \cdot u_{x_j}\, dx$$

Show that this quantity decreases in time when $n = 2$.