# DISPERSIVE PDE 23, PROBLEM SET 2

(1) Check carefully that if $f, g \in \mathcal{S}(\mathbb{R}^n)$ (Schwartz functions), then the formula (1. 7) in lecture1.pdf furnishes a function $\phi(t,x) \in C^\infty(\mathbb{R}^{1+n})$ and such that
$$\phi(t,\cdot) \in \mathcal{S}(\mathbb{R}^n),\ \forall t \in \mathbb{R}.$$
(2) Carefully check the limiting relation
$$\lim_{\epsilon \downarrow 0} K_\epsilon(r) = (2\pi)^{-1} t^{-\frac{1}{2}} e^{i\frac{r^2}{4t}} \cdot \int_{\mathbb{R}} e^{-iz^2}\, dz$$
in the proof of Lemma 2.1 in lecture1.pdf

(3) Give the details in the proof that
$$\int_{\mathbb{R}} e^{-iz^2}\, dz = \sqrt{\pi} \cdot \frac{1-i}{\sqrt{2}}.$$
(4) Derive the solution formula for the heat equation: the initial value problem
$$\psi_t = \psi_{xx},\ \psi(0,x) = f(x) \in \mathcal{S}(\mathbb{R}),$$
is solved for $t > 0$ by the formula
$$\psi(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y)\, dy,$$
and $\psi$ is in $C^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap C^0(\overline{\mathbb{R}}_+ \times \mathbb{R})$.

(5) (*) Show that if $f \in C_0^\infty(\mathbb{R}) \backslash \{0\}$, then the solution of
$$i\psi_t + \psi_{xx} = 0,\ \psi(0,x) = f(x)$$
satisfies $\psi(t,\cdot) \notin C_0^\infty(\mathbb{R})$ for all $t \in \mathbb{R}\backslash\{0\}$.