GLOBAL WELL-POSEDNESS FOR ENERGY SUB CRITICAL NONLINEAR WAVE EQUATIONS

Here we consider the question whether we can construct *large global solutions* for some 'sufficiently nice' nonlinear wave equations, even in dimensions $n > 1$. To be specific, we shall consider the case $n = 3$. The examples we consider have two crucial features:

- Hamiltonian structure: they have a conserved energy, which is positive definite.
- They are energy sub critical. In fact, this feature will imply for these examples that they are strongly locally well-posed even slightly below energy class data.

Combining these two features will easily allow us to conclude the global well-posedness for large data for these models. We shall in effect use two distinct methods: one will involve the use of Strichartz estimates to infer the local well-posedness at energy regularity, The other method will involve use of the classical Kirchhoff parametrix.

1. The energy sub-critical defocussing nonlinear wave equation

We now fix the background to be $\mathbb{R}^{3+1}$, and we consider the following family of nonlinear wave equations:

$$-u_{tt} + \triangle u = |u|^{p-1} \cdot u. \tag{1.1}$$

For technical reasons we shall restrict to $p > 1$, which then implies that the function

$$F(u) := |u|^{p-1} \cdot u$$

is of class C^1, and satisfies the bound

$$\big|F(u) - F(v)\big| \leq C_p \cdot |u - v| \cdot \big(|u|^{p-1} + |v|^{p-1}\big). \tag{1.2}$$

We shall restrict to real-valued solutions throughout. Then we have

Definition 1.1. *The model* (1.1) *with* $1 < p < 5$ *is called the defocussing energy sub-critical nonlinear wave equation on* $\mathbb{R}^{3+1}$.

The name *defocussing* comes from the positive sign of the nonlinearity; the same equation with a $-$-sign would be called *focussing*. The basic distinction for the powers p under consideration here is that solutions for the focussing problem tend to blow up, while those for the defocusing problem remain regular. This in turn is linked to the fact that the preserved energy

$$E := \int_{\mathbb{R}^3} \big(\frac{1}{2}|\nabla_{t,x} u|^2 + \frac{1}{p+1}|u|^{p+1}\big)\, dx \tag{1.3}$$

is positive in the defocussing case.

Precisely, consider the following initial value problem

$$-u_{tt} + \triangle u = |u|^{p-1} \cdot u,\ u[0] = \big(u,\, u_t\big)|_{t=0} = \big(u_0, u_1\big). \tag{1.4}$$

Then we have the

Theorem 1.2. *The problem* (1.4) *with* $1 \leq p < 5$ *is globally well-posed in* $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$*. Precisely, given*

$$u[0] \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3),$$

there is a unique global in time solution $u(t, \cdot)$ *of class* $C^0\big(\mathbb{R}; H^1(\mathbb{R}^3)\big) \cap C^1\big(\mathbb{R}; L^2(\mathbb{R}^3)\big)$.

We note that for example in the case $p = 3$, we can also conclude that if $u[0] \in H^s \times H^{s-1}$, $s \geq 1$, then the solution will be automatically of this higher regularity. In particular, one concludes for this case that if the data are C^∞, then so is the solution.

2. The approach via Strichartz estimates

We recall first from an earlier lecture that we have the following Strichartz estimate for *free waves* on $\mathbb{R}^{3+1}$:

$$\|u\|_{L_t^5 L_x^{\frac{10}{3}}} \leq C \cdot \|u[0]\|_{\dot{H}^{\frac{2}{5}} \times \dot{H}^{-\frac{3}{5}}}.$$

Assuming for the rest of this section[1] that $3 \leq p < 5$, then we obtain by using the Sobolev embedding that

$$\|u\|_{L_t^5 L_x^{2p}} \leq C \cdot \|u[0]\|_{\dot{H}^s \times \dot{H}^{s-1}}, \tag{2.1}$$

where we have

$$s = s(p) = \frac{2}{5} + 3 \cdot \left(\frac{3}{10} - \frac{1}{2p}\right) = \frac{13}{10} - \frac{3}{2p} \in (0,1).$$

In particular, we have $s = 1$ for $p = 5$.
The preceding bound is useful in the high-frequency regime were $|\xi| \geq 1$, where we can further deduce that

$$\|P_{\geq 1} u\|_{L_t^5 L_x^{2p}} \leq C \cdot \|P_{\geq 1} u[0]\|_{\dot{H}^s \times \dot{H}^{s-1}} \leq C' \cdot \|u[0]\|_{\dot{H}^1 \times L^2}. \tag{2.2}$$

In the low frequency regime $|\xi| < 1$ we can simply use the following estimate, again for free waves on $\mathbb{R}^{3+1}$:

$$\|P_{<1} u\|_{L_t^\infty L_x^{2p}} \leq C \cdot \|u[0]\|_{\dot{H}^1 \times L^2}. \tag{2.3}$$

In fact, this is simply a consequence of the Sobolev embedding. If we restrict to time intervals I of length $|I| \leq 1$, centered around $t = 0$, then we can use Holder's inequality to deduce

$$\|P_{<1} u\|_{L_t^5 L_x^{2p}(I \times \mathbb{R}^3)} \leq C \cdot \|u[0]\|_{\dot{H}^1 \times L^2}.$$

We can then combine the high frequency and the low frequency bound to state

$$\|u\|_{L_t^5 L_x^{2p}(I \times \mathbb{R}^3)} \leq C \cdot \|u[0]\|_{\dot{H}^1 \times L^2}. \tag{2.4}$$

Introduce the norm

$$\|u\|_X := \|u\|_{L_t^5 L_x^{2p}(I \times \mathbb{R}^3)} + \|\nabla_{t,x} u\|_{L_t^\infty L_x^2(I \times \mathbb{R}^3)}.$$

Then we observe

Lemma 2.1. *Denoting*

$$T(u) := S(t)(f,g) + \int_0^t U(t-s)\big(|u|^{p-1}u\big)(s,\cdot)\,ds,$$

we have the bounds

$$\|T(u)\|_X \leq C \cdot \big(\|(f,g)\|_{\dot{H}^1 \times L^2} + |I|^\alpha \cdot \|u\|_X^p\big)$$

We also have the estimate

$$\|T(u) - T(v)\|_X \leq C \cdot |I|^\alpha \cdot \|u - v\|_X.$$

for a constant $\alpha = \alpha(p) > 0$.

Proof. We prove the first estimate, the second being similar. From various applications of the triangle inequality, we deduce that

$$\begin{aligned} \Big\| \int_0^t U(t-s)\big(|u|^{p-1}u\big)(s,\cdot)\,ds \Big\|_{L_t^5 L_x^{2p}(I \times \mathbb{R}^3)} &\leq \Big\| \int_0^t \big|U(t-s)\big(|u|^{p-1}u\big)(s,\cdot)\big|\,ds \Big\|_{L_t^5 L_x^{2p}(I \times \mathbb{R}^3)} \\ \leq \Big\| \int_I \big|U(t-s)\big(|u|^{p-1}u\big)(s,\cdot)\big|\,ds \Big\|_{L_t^5 L_x^{2p}(I \times \mathbb{R}^3)} &\leq \int_I \big\|U(t-s)\big(|u|^{p-1}u\big)(s,\cdot)\big\|_{L_t^5 L_x^{2p}(I \times \mathbb{R}^3)}\,ds. \end{aligned}$$

Further from estimate (2.4) we find

$$\int_I \big\|U(t-s)\big(|u|^{p-1}u\big)(s,\cdot)\big\|_{L_t^5 L_x^{2p}(I \times \mathbb{R}^3)}\,ds \leq C \cdot \big\||u|^{p-1}u\big\|_{L_t^1 L_x^2(I \times \mathbb{R}^3)}$$

[1] This is just for technical convenience.

Finally using Holder's inequality, we find that

$$\big\||u|^{p-1}u\big\|_{L_t^1 L_x^2(I\times\mathbb{R}^3)} \le |I|^{\frac{1}{p}-\frac{1}{5}} \cdot \big\|u\big\|_X^p,$$

whence it suffices to set $\alpha = \frac{1}{p} - \frac{1}{5}$. The energy norm in $\|u\|_X$ is controlled similarly. □

Using a standard fixed point argument, we then deduce the following

Proposition 2.2. *Letting* $(u_0, u_1) \in \dot{H}^1 \times L^2$*,m there is*

$$T = T(\big\|(u_0, u_1)\big\|_{\dot{H}^1\times L^2} > 0$$

with the property that (1.4) *admits a solution on* $(-T, T) \times \mathbb{R}^3)$ *of class* $C^0\big(\mathbb{R}; \dot{H}^1(\mathbb{R}^3)\big) \cap C^1\big(\mathbb{R}; L^2(\mathbb{R}^3)\big)$*. Moreover, this solution satisfies*

$$\big\|u\big\|_X \le C \cdot \big\|(u_0, u_1)\big\|_{\dot{H}^1\times L^2},$$

where $I = (-T, T)$*.*

The proof of Theorem 1.2 is then completed via the following

Lemma 2.3. *Let* $u \in C^0\big(J; \dot{H}^1(\mathbb{R}^3)\big) \cap C^1\big(J; L^2(\mathbb{R}^3)\big)$ *be a solution of* (1.4) *on* $J \times \mathbb{R}^3$*, where* $J = (-T, T)$ *is an open time interval. Also assume that* $u(0, \cdot) \in H^1(\mathbb{R}^3)$*. Then the energy* (1.3) *is well-defined for any* $t \in J$ *and in fact independent of time.*

Proof. (sketch) Let us assume $p = 3$, where the nonlinearity is actually given by $|u|^2 \cdot u = u^3$, and the energy is given by

$$E = \int_{\mathbb{R}^3} \Big[\frac{1}{2}|\nabla_{t,x}u|^2 + \frac{1}{4}u^4\Big]\, dx.$$

That this is well-defined follows from the fact that $u(t, \cdot) \in L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$ for any $t \in J$, and the fact that

$$\big\|u\big\|_{L^4} \le \big\|u\big\|_{L^2}^{\frac{1}{4}} \cdot \big\|u\big\|_{L^6}^{\frac{3}{4}}.$$

That $u(t, \cdot) \in L^6$ follows from $u(t, \cdot) \in \dot{H}^1$ and the Sobolev embedding. Further we observe that

$$u(t, \cdot) = \int_0^t u_t(s, \cdot)\, ds + u(0, \cdot),$$

and so since $u(0, \cdot) \in L^2$ by assumption, we also get $u(t, \cdot) \in L^2$ by hypothesis, for any $t \in J$.
It remains to check that E is in fact time independent. For this we first formally compute

$$\begin{aligned}\frac{d}{dt}E &= \int_{\mathbb{R}^3} \big(u_{tt} \cdot u_t + \nabla_x u_t \cdot \nabla_x u + u^3 \cdot u_t\big)\, dx \\ &= \int_{\mathbb{R}^3} u_t \cdot \big(u_{tt} - \triangle u + u^3\big)\, dx = 0.\end{aligned}$$

In order to make the last step rigorous, it suffices for example to first assume that $u(t, \cdot) \in C_0^\infty(\mathbb{R}^3)$, which can be achieved by approximating the data $u[0]$ by data in $C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$, and then passing to the limit. To implement this rigorously, one can use the preceding lemma. The case of general p can be handled similarly. □

Completion of the proof of Theorem 1.2: Starting at time $t = 0$, we use Proposition 2.2 to construct $T = T(\big\|(u_0, u_1)\big\|_{\dot{H}^1\times L^2}) > 0$ such that a solution exists on $[-T, T] \times \mathbb{R}^3$. The preceding lemma implies that

$$\big\|u[\pm T]\big\|_{\dot{H}^1\times L^2} \le C \cdot E^{\frac{1}{2}}.$$

We can then again apply Proposition 2.2 with time 0 replaced by $\pm T$ and prolong the solution to a larger time interval. Again the uniform energy bound applies there, and so we can continue prolonging the solution until it covers all of the time axis.

3. The approach via the Kirchhoff parametrix

Here we describe a very different approach to the same problem, but which relies on tools *from the very beginning of this course*. The idea is to derive an *a priori bound on the L^∞-norm* of a solution near a putative singularity, and thereby showing that the solution can be extended. To begin with, we have a basic local well-posedness result which can be derived by simple energy methods and the standard Sobolev embedding. As we have the much sharper Proposition 2.2, we only state this to emphasize that its proof does not rely on Strichartz estimates:

Lemma 3.1. *Let $p > 2$. Then the problem (1.4) on $\mathbb{R}^{3+1}$ is locally well-posed for $u[0] \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$.*

We next deduce a *breakdown criterion* in terms of the L^∞-norm of the solution:

Lemma 3.2. *Let $u \in C^0\big([0,T); H^2(\mathbb{R}^3)\big) \cap C^1\big([0,T); H^1(\mathbb{R}^3)\big)$ a local solution of (1.4) with $p > 2$, $0 < T < +\infty$. Assume that*

$$\sup_{t<T} \big\|u(t,\cdot)\big\|_{L^\infty(\mathbb{R}^3)} < \infty.$$

Then the limits

$$\lim_{t\to T} u(t,\cdot),\ \lim_{t\to T} u_t(t,\cdot),$$

exist in $H^2(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$, respectively. In particular, the solution u can be extended beyond time T.

Proof. First, we note that since

$$u(t,\cdot) - u(0,\cdot) = \int_0^t u_t(s,\cdot)\,ds,$$

for $0 \le t < T$, and

$$\|u_t\|^2_{L^2(\mathbb{R}^3)} \le 2E,$$

where E is the preserved energy which is a priori finite, we easily deduce that

$$\lim_{t\to T} u(t,\cdot)$$

exists in $L^2(\mathbb{R}^3)$. Our a priori bound on $\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^3)}$ then implies that

$$\lim_{T'\to T} \big\||u|^{p-1}u(t,\cdot)\big\|_{L^1_t L^2_x([0,T']\times\mathbb{R}^3)}$$

exists. Using that

$$\nabla_x u(t,\cdot) = \nabla_x S(t)\big(u_0,u_1\big) + \nabla_x \int_0^t U(t-s)\big(|u|^{p-1}u\big)(s,\cdot)\,ds,$$

and that

$$\begin{aligned}\big\|\nabla_x U(t-s)P_{>K}\big(|u|^{p-1}u\big)(s,\cdot)\big\|_{L^2_x} &\le C\cdot K^{-1}\cdot \big\|u\big\|^{p-1}_{L^\infty}\cdot\big\|\nabla_x u\big\|_{L^2_x}\\ &\le C\cdot K^{-1}\cdot(2E)^{\frac12},\end{aligned}$$

one deduces (*exercise*) that for given $\varepsilon > 0$, we have

$$\big\|\nabla_x u(t,\cdot) - \nabla_x u(t',\cdot)\big\|_{L^2_x} < \varepsilon$$

provided $t, t' \in [0,T)$ satisfy $|t-t'| < \delta(\varepsilon)$. In turn this yields that

$$\lim_{t\to T} \nabla_x u(t,\cdot)$$

exists in L^2, and one argues similarly for $\lim_{t\to T}\partial_t u(t,\cdot)$. Finally, we consider

$$\nabla^2_x u(t,\cdot) = \nabla^2_x S(t)\big(u_0,u_1\big) + \nabla_x \int_0^t U(t-s)\big(\nabla_x\big[|u|^{p-1}u\big]\big)(s,\cdot)\,ds.$$

Using the preceding, we infer that

$$\limsup_{t\to T} \big\|\nabla^2_x u(t,\cdot)\big\|_{L^2_x} < +\infty,$$

which then implies that

$$\limsup_{t<T} \big\|\nabla_x^2\big[|u|^{p-1}u(t,\cdot)\big]\big\|_{L_x^2} < +\infty.$$

In turn, we easily infer that

$$\big\|\nabla_x U(t-s)P_{>K}\big(\nabla_x\big[|u|^{p-1}u\big]\big)(s,\cdot)\big\|_{L_x^2} \leq C\cdot K^{-1}\cdot \big\|u\big\|_{H^2}^p.$$

Using the Duhamel formula for $\nabla_x^2 u(t,\cdot)$ one then derives that the limit of $\nabla_x^2 u(t,\cdot)$ exists in L_x^2 as $t\to T$. □

We now reduce to proving the following

Proposition 3.3. *Let $0<T<+\infty$ and assume that $u\in C^0\big([0,T);H^2(\mathbb{R}^3)\big)\cap C^1\big([0,T);H^1(\mathbb{R}^3)\big)$ solves (1.1) with $2<p<5$. Then*

$$u\in L^\infty\big([0,T)\times\mathbb{R}^3\big).$$

Proof. Pick any point $(t_*,x_*)\in\mathbb{R}^{3+1}$ with $t_*<T$ but close to T. We may assume that $x_*=0$. Then we use the Kirchhoff formula for the inhomogeneous wave equation to conclude that with $T'<t_*<T$

$$u(t_*,0) = S(t_*-T')\big(u[T']\big)(0) + \int_{T'}^{t_*}\frac{1}{4\pi(t_*-s)}\cdot\int_{|x|=t_*-s}|u|^{p-1}u(s,x)\,d\sigma ds \tag{3.1}$$

We know that

$$\big\|S(t_*-T')\big(u[T']\big)(0)\big\|_{H^2}\leq C\cdot\big\|u[T']\big\|_{H^2\times H^1},$$

and so we obtain an L^∞-bound for this contribution using Sobolev's embedding. The main point is to estimate the double integral term, which is over the backward light cone centered at $(t_*,0)$ and truncated at $t=T'$. In fact, let us call the solid light cone

$$C := \{(t,x)\in\mathbb{R}^{3+1},\ |t-t_*|\geq|x|,\ t\in[T',t_*]\},$$

and let us denote its mantel by

$$M := \{(t,x)\in\mathbb{R}^{3+1},\ |t-t_*|=|x|,\ t\in[T',t_*]\}.$$

Then we have the following crude estimate for the double integral:

$$\Big|\int_{T'}^{t_*}\frac{1}{4\pi(t_*-s)}\cdot\int_{|x|=t_*-s}|u|^{p-1}u(s,x)\,d\sigma ds\Big| \leq \big\|u\big\|_{L^\infty(C)}\cdot\Big|\int_{T'}^{t_*}\frac{1}{4\pi(t_*-s)}\cdot\int_{|x|=t_*-s}|u|^{p-1}(s,x)\,d\sigma ds\Big|.$$

The trick now shall be to use the local form of energy conservation, and specifically the most precise variant of Proposition 1.2 from lecture2.pdf, to deduce an a priori bound on the remaining integral term. Recalling the proof of this proposition, which is easily adapted to the nonlinear problem at hand (see the preceding section), we have the *local energy identity*

$$\begin{aligned}&\int_{|x|\leq|t_*-T'|}\big(|\frac{1}{2}|\nabla_{t,x}u|^2+\frac{1}{p+1}|u|^{p+1}\big)\,dx\\ &=\int_M\big(\frac{1}{2}\big(u_t-\omega\cdot\nabla_x u\big)^2+\frac{1}{2}\big(|\nabla_x u|^2-(\omega\cdot\nabla_x u)^2\big)+\frac{1}{p+1}|u|^{p+1}\big)\,d\sigma,\end{aligned} \tag{3.2}$$

where we parametrise the mantel via $(s,\omega)\in[T',t_*]\times S^2\longrightarrow(s,(t_*-s)\cdot\omega)$. Now the key is to observe that the derivative

$$\partial_t-\omega\cdot\nabla_x,$$

as well as the 'angular derivatives

$$\omega_i\partial_{x_j}-\omega_j\cdot\partial_{x_i},\ 1\leq i,j\leq 3,$$

generate the derivatives tangential to the mantel M. Since M is a 3-dimensional smooth manifold, we can apply the Sobolev embedding to deduce that

$$\big\|u\big\|_{L^6(M)}\leq C_1\cdot\big(\int_{\mathbb{R}^3}\big(|\frac{1}{2}|\nabla_{t,x}u|^2+\frac{1}{p+1}|u|^{p+1}\big)\,dx\big)^{\frac{1}{2}} = C_2\cdot E^{\frac{1}{2}}.$$

The energy here is of course bounded due to our assumptions, and time independent.
Coming back to the integral above, we now use Holder's inequality to find that

$$\begin{aligned}
&\Big|\int_{|x|=t_*-s}|u|^{p-1}(s,x)\,d\sigma\Big| = \big\|u\big\|^{p-1}_{L^{p-1}(|x|=t_*-s)}\\
&\leq \big(\big\|u\big\|_{L^6(|x|=t_*-s)}\cdot\big\|1\big\|_{L^{\frac{6(p-1)}{7-p}}(|x|=t_*-s)}\big)^{p-1}\\
&= \big\|u\big\|^{p-1}_{L^6(|x|=t_*-s)}\cdot(t_*-s)^{\frac{7-p}{3}}.
\end{aligned}$$

We can then conclude via another application of Holder's inequality that

$$\begin{aligned}
&\Big|\int_{T'}^{t_*}\frac{1}{4\pi(t_*-s)}\cdot\int_{|x|=t_*-s}|u|^{p-1}(s,x)\,d\sigma ds\Big|\\
&\leq \int_{T'}^{t_*}\frac{1}{4\pi(t_*-s)}\cdot\big\|u\big\|^{p-1}_{L^6(|x|=t_*-s)}\cdot(t_*-s)^{\frac{7-p}{3}}\,ds\\
&\leq \big\|u\big\|^{p-1}_{L^6(M)}\cdot\Big(\int_{T'}^{t_*}(t_*-s)^{\frac{2(4-p)}{7-p}}\,ds\Big)^{\frac{7-p}{6}}.
\end{aligned}$$

Since

$$\frac{2(4-p)}{7-p} > -1$$

for $2 < p < 5$, and since we also have the bound $\big\|u\big\|_{L^6(M)} \leq C_2\cdot E^{\frac{1}{2}}$, we can choose T' sufficiently close to T and such that

$$\Big|\int_{T'}^{t_*}\frac{1}{4\pi(t_*-s)}\cdot\int_{|x|=t_*-s}|u|^{p-1}(s,x)\,d\sigma ds\Big| < \frac{1}{2}.$$

This then gives us the desired bound

$$\Big|\int_{T'}^{t_*}\frac{1}{4\pi(t_*-s)}\cdot\int_{|x|=t_*-s}|u|^{p-1}u(s,x)\,d\sigma ds\Big| \leq \frac{1}{2}\cdot\big\|u\big\|_{L^\infty(C)}.$$

Coming back to (3.1), we can replace the point $(t_*, 0)$ by any other point inside C to infer the same estimate, and so we have now proved that

$$\big\|u\big\|_{L^\infty(C)} \leq C_3\cdot\big\|u[T']\big\|_{H^2\times H^1} + \frac{1}{2}\cdot\big\|u\big\|_{L^\infty(C)}.$$

The estimate

$$\big\|u\big\|_{L^\infty(C)} \leq 2C_3\cdot\big\|u[T']\big\|_{H^2\times H^1}$$

follows.
But the cone C was chosen arbitrarily (it only needs to have a base sufficiently close to the time slice $t = T$), and so the preceding bound implies an a priori bound on $\|u\|_{L^\infty([T',T]\times\mathbb{R}^3)}$. □

4. Concluding remarks

The preceding proof is essentially that of Konrad Joergens from 1961. The *defocussing energy critical case*, corresponding to $p = 5$ and $n = 3$, was resolved by Grillakis for arbitrary large initial data in 1990. Nobody knows at this point what happens in the so-called *energy super critical defocussing regime*, corresponding to $p > 5$ when $n = 3$, and without any size restrictions on the data.