IMPROVED LOCAL WELL-POSEDNESS FOR NONLINEAR WAVE EQUATIONS

In this lecture we shall take advantage of the new estimates derived in the last lecture to deduce a much better well-posedness result for a certain nonlinear wave equation. To begin with we quickly recall what the energy method gives in terms of local well-posedness, and will then see that Strichartz estimates allow us to push this down by half a degree of regularity. What is most remarkable about this example shall be the fact that we arrive at a *sharp local well-posedness result*, in the sense that it cannot be improved. This shows the fundamental significance of the Strichartz estimates.

1. The model and what the energy method gives

For simplicity we shall restrict ourselves to the physical space time $\mathbb{R}^{1+3}$, and we shall consider the following model:

$$\Box u = -u_{tt} + \triangle u = u_t^2 \tag{1.1}$$

We note right away that this model is not Hamiltonian, i. e. it does not come with a conserved energy. Thus it is not particularly physical, but this will not concern us. In order to implement the energy method, we need to control the L^∞-norm of u_t. This can be achieved via Sobolev:

$$\|u_t\|_{L^\infty(\mathbb{R}^n)} \lesssim \|u_t\|_{H^s(\mathbb{R}^n)},\ s > \frac{n}{2}.$$

Since there is already a derivative ∂_t in u_t, this suggests that we should aim for well-posedness in H^s with $s > \frac{n}{2} + 1$, i. e. for $n = 3$ this method ought to give local well-posedness in H^s with $s > \frac{5}{2}$. In fact, we have the following result:

Proposition 1.1. *The model* (1.1) *is locally well-posed in* $H^s(\mathbb{R}^3)$ *for* $s > \frac{5}{2}$. *Thus given*

$$(f, g) \in H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3),\ s > \frac{5}{2},$$

there is $T = T(\|(f, g)\|_{H^s(\mathbb{R}^3)\times H^{s-1}(\mathbb{R}^3)}) > 0$ *and a unique solution*

$$u \in C^0([-T, T], H^s(\mathbb{R}^3)) \cap C^1([-T, T], H^{s-1}(\mathbb{R}^3))$$

of (1.1) *with*

$$(u, u_t)|_{t=0} = (f, g).$$

Proof. It is similar to the one of Theorem 2. 1 in lecture 5 and we shall not give all details. To begin with, we recall from Prop. 2.4 from lecture 4 that the solution of

$$\Box\psi = F,\ \psi[0] = (\psi(0, \cdot), \psi_t(0, \cdot)) = (f(\cdot), g(\cdot))$$

is given by the Duhamel formula

$$\psi(t, \cdot) = S(t)(f, g) + \int_0^t U(t - s)F(s)\, ds$$

where we have

$$\big(U(t)F\big)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_0^t e^{ix\cdot\xi} \cdot \frac{\sin[(t - s)|\xi|]}{|\xi|} \widehat{F}(s, \xi)\, dsd\xi.$$

Observe that

$$\partial_t\big(U(t)F\big)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_0^t e^{ix\cdot\xi} \cdot \cos[(t - s)|\xi|]\widehat{F}(s, \xi)\, dsd\xi$$

It is then easily seen that

$$\|\psi_t(t, \cdot)\|_{H^{s-1}(\mathbb{R}^n)} \le \|(f, g)\|_{H^s(\mathbb{R}^n)\times H^{s-1}(\mathbb{R}^n)} + \|F\|_{L_t^1 H^{s-1}(\mathbb{R}^n)}.$$

In order to solve (1.1) (for $n = 3$) we use a simple fixed point argument in the space

$$X := C^0([-T,T], H^s(\mathbb{R}^3)) \cap C^1([-T,T], H^{s-1}(\mathbb{R}^3)),\ s > \frac{5}{2}.$$

This is a Banach space when equipped with the norm

$$\|u\|_X := \|u\|_{L_t^\infty H^s([-T,T]\times\mathbb{R}^3)} + \|u_t\|_{L_t^\infty H^{s-1}([-T,T]\times\mathbb{R}^3)}$$

Taking advantage of Lemma 2.2 from lecture 5, we deduce that

$$\|u_t^2\|_{L_t^\infty H^{s-1}([-T,T]\times\mathbb{R}^3)} \leq C \cdot \|u_t\|^2_{L_t^\infty H^{s-1}([-T,T]\times\mathbb{R}^3)} \leq C_1 \cdot \|u\|_X^2.$$

If we then define the map

$$T(\psi) = S(t)(f,g) + \int_0^t U(t-s)(\psi_s^2(s,\cdot))\,ds,$$

we have the mapping bound

$$\|T(\psi)\|_X \leq C_2 \cdot \|(f,g)\|_{H^s\times H^{s-1}} + C_3 \cdot T \cdot \|\psi\|_X^2.$$

This map is a contraction on

$$B_{2C_2\|(f,g)\|_{H^s\times H^{s-1}}}(0) \subset X$$

for $T = T(\|(f,g)\|_{H^s\times H^{s-1}}) > 0$ small enough, which implies the proposition via the Banach fixed point theorem. □

2. The effect of Strichartz estimates

Our next goal is to lower the regularity requirements by using the wave Strichartz estimates available in dimension $n = 3$. This shall enable us to prove the following

Theorem 2.1. *(Ponce-Sideris 1993) Let $s > 2$. Then given $(f,g) \in H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$, there is*

$$T = T(\|(f,g)\|_{H^s(\mathbb{R}^3)\times H^{s-1}(\mathbb{R}^3)}) > 0$$

such that the problem (1.1) *with initial data (f,g) at time $t = 0$ admits a unique solution*

$$u \in C^0\big([-T,T], H^s(\mathbb{R}^3)\big) \cap C^1\big([-T,T], H^{s-1}(\mathbb{R}^3)\big).$$

Proof. This is again based on a fixed point argument, but the space we shall use to contain u will be more complicated, and based on Strichartz estimates. Given $s > 2$, which we may assume is in $(2, \frac{5}{2}]$, pick $p > 2$ such that

$$s > \frac{5}{2} - \frac{1}{p}.$$

Recall from lecture 7 that if M satisfies $\frac{1}{p} + \frac{1}{M} = \frac{1}{2}$ then (p, M) is sharp wave Strichartz admissible on $\mathbb{R}^{1+3}$. Then if u is a frequency 1 free wave with data $u[0] = (f,g)$, we have the bound

$$\|u\|_{L_t^p L_x^M(\mathbb{R}^{1+3})} \leq C \cdot \|(f,g)\|_{L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)},$$

and thanks to the fact that for a frequency 1 function h on $\mathbb{R}^3$ we have

$$\|h\|_{L_x^\infty(\mathbb{R}^3)} \leq C_1 \cdot \|h\|_{L_x^M(\mathbb{R}^3)},$$

we also infer that

$$\|u\|_{L_t^p L_x^\infty(\mathbb{R}^{1+3})} \leq C \cdot \|(f,g)\|_{L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)},$$

Re-scaling the wave u to frequency $\lambda \in 2^{\mathbb{Z}}$ as we did in lecture 7, we then infer the bound

$$\|u_\lambda\|_{L_t^p L_x^\infty(\mathbb{R}^{1+3})} \leq C \cdot \lambda^{\frac{3}{2}-\frac{1}{p}} \|(f_\lambda, g_\lambda)\|_{L^2(\mathbb{R}^3)\times \dot{H}^{-1}(\mathbb{R}^3)},$$

and differentiating with respect to time t we also infer that

$$\|\partial_t u_\lambda\|_{L_t^p L_x^\infty(\mathbb{R}^{1+3})} \leq C \cdot \lambda^{\frac{5}{2}-\frac{1}{p}} \|(f_\lambda, g_\lambda)\|_{L^2(\mathbb{R}^3)\times \dot{H}^{-1}(\mathbb{R}^3)}.$$

If we then set $\lambda = 2^k$ and sum over all $k \in \mathbb{Z}$, we find that for a general free wave on $\mathbb{R}^{1+3}$ (not frequency localized) we have

$$\sum_k \|P_k u_t\|_{L_t^p L_x^\infty(\mathbb{R}^{1+3})} \leq C \cdot \sum_k 2^{(\frac{5}{2}-\frac{1}{p})\cdot k} \cdot \|(P_k f, P_k g)\|_{L^2(\mathbb{R}^3)\times \dot{H}^{-1}(\mathbb{R}^3)}.$$

Since $2 < \frac{5}{2} - \frac{1}{p} < s$, an application of the Cauchy-Schwarz inequality yields (exercise!)

$$\sum_k 2^{(\frac{5}{2}-\frac{1}{p})\cdot k} \cdot \|(P_k f, P_k g)\|_{L^2(\mathbb{R}^3)\times \dot{H}^{-1}(\mathbb{R}^3)} \leq C_1 \cdot \|(f,g)\|_{H^s(\mathbb{R}^3)\times H^{s-1}(\mathbb{R}^3)}$$

for a constant C_1 depending only on p, s. Setting

$$s_1 := \frac{3}{2} - \frac{1}{p},$$

we shall establish existence of the desired solution for (1.1) by working with the norm

$$\|u\|_X := \sum_k \|P_k u_t\|_{L_t^p L_x^\infty([-T,T]\times\mathbb{R}^3)} + \big(\sum_k \|\nabla_{t,x} P_k u\|^2_{L_t^\infty H^{s-1}([-T,T]\times\mathbb{R}^3))}\big)^{\frac{1}{2}}.$$

To begin with, we shall need a fractional derivative type Leibniz rule, as in

Lemma 2.2. *For p, s as before, we have the estimate*

$$\sum_k 2^{s_1 k} \cdot \|P_k(u_t^2)\|_{L_t^1 L_x^2([-T,T]\times\mathbb{R}^3)} \leq C_1 \cdot T^{\frac{1}{p'}} \cdot \|u\|_X^2,$$
$$\big(\sum_k \|P_k(u_t^2)\|^2_{L_t^1 H_x^{s-1}([-T,T]\times\mathbb{R}^3)}\big)^{\frac{1}{2}} \leq C_2 \cdot T^{\frac{1}{p'}} \cdot \|u\|_X^2,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. (lemma) It suffices to show that

$$\sum_k 2^{s_1 k} \cdot \|P_k(u_t^2)\|_{L_t^p L_x^2([-T,T]\times\mathbb{R}^3)} \leq C_1 \cdot \|u\|_X^2,$$
$$\big(\sum_k \|P_k(u_t^2)\|^2_{L_t^p H_x^{s-1}([-T,T]\times\mathbb{R}^3)}\big)^{\frac{1}{2}} \leq C_2 \cdot \|u\|_X^2,$$

since the lemma follows then by application of Holder's inequality. The first of these bounds is obtained by applying a Littlewood-Paley decomposition like we did in lecture 5. Thus write

$$P_k(u_t^2) = P_k\big(P_{<k-10}(u_t)\cdot u_t\big) + P_k\big(P_{[k-10,k+10]}(u_t)\cdot u_t\big) + P_k\big(P_{>k-10}(u_t)\cdot u_t\big).$$

Then the strategy is to place the *high-frequency factor* in each expression into $L_t^\infty H^{s_1}$, where we crucially note that $s_1 < s-1$. We give details for the first term on the right, leaving the other two as exercises. Observe that

$$P_k\big(P_{<k-10}(u_t)\cdot u_t\big) = P_k\big(P_{<k-10}(u_t)\cdot P_{[k-5,k+5]}u_t\big).$$

Using Holder's inequality, we can bound

$$\Big\|P_k\big(P_{<k-10}(u_t)\cdot P_{[k-5,k+5]}u_t\big)\Big\|_{L_t^p L_x^2([-T,T]\times\mathbb{R}^3)} \leq \|P_{<k-10}(u_t)\|_{L_t^p L_x^\infty([-T,T]\times\mathbb{R}^3)} \cdot \|P_{[k-5,k+5]}u_t\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R}^3)}$$

But then we can estimate

$$\|P_{<k-10}(u_t)\|_{L_t^p L_x^\infty([-T,T]\times\mathbb{R}^3)} \leq \sum_{k_1<k-10} \|P_{k_1}(u_t)\|_{L_t^p L_x^\infty([-T,T]\times\mathbb{R}^3)} \leq \|u\|_X.$$

Further, taking advantage of the fact that $s_1 < s-1$ and again using the Cauchy-Schwarz inequality, we find that

$$\sum_k 2^{s_1 k} \cdot \|P_{[k-5,k+5]}u_t\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R}^3)} \leq C_3 \cdot \big(\sum_k \|P_k u_t\|^2_{L_t^\infty H_x^{s-1}([-T,T]\times\mathbb{R}^3)}\big)^{\frac{1}{2}} \leq C_3 \cdot \|u\|_X.$$

We conclude that

$$\sum_k 2^{s_1\cdot k}\cdot \big\|P_k\big(P_{<k-10}(u_t)\cdot P_{[k-5,k+5]}u_t\big)\big\|_{L_t^pL_x^2([-T,T]\times\mathbb{R}^3)}\leq C_4\cdot \big\|u\big\|_X^2.$$

To obtain the second bound stated at the beginning of the proof of the lemma, we proceed similarly, and use for example

$$\big\|P_{<k-10}u_t\cdot P_{k-5,k+5]}u_t\big\|_{L_t^pH_x^{s-1}([-T,T]\times\mathbb{R}^3)}\leq C_5\cdot \big\|P_{<k-10}u_t\big\|_{L_t^pL_x^\infty([-T,T]\times\mathbb{R}^3)}\cdot \big\|P_{[k-5,k+5]}u_t\big\|_{L_t^\infty H_x^{s-1}([-T,T]\times\mathbb{R}^3)}.$$

Since

$$\big\|u_t\big\|_{L_t^pL_x^\infty([-T,T]\times\mathbb{R}^3)}\leq \sum_k \big\|P_ku_t\big\|_{L_t^pL_x^\infty([-T,T]\times\mathbb{R}^3)}\leq \big\|u\big\|_X,$$

we find that

$$\big(\sum_k \big\|P_{<k-10}u_t\cdot P_{[k-5,k+5]}u_t\big\|^2_{L_t^pH_x^{s-1}([-T,T]\times\mathbb{R}^3)}\big)^{\frac12}\leq C_5\cdot \big\|u\big\|_X^2.$$

□

Continuing with the proof of the theorem, we look for a fixed point of the mapping

$$u\to Tu(t,\cdot):=S(t)(f,g)+\int_0^t U(t-s)\big(u_t^2\big)\,ds$$

in the space X. Thanks to the considerations preceding the previous lemma, we have the bound

$$\big\|S(t)(f,g)\big\|_X\leq C_6\cdot \big\|(f,g)\big\|_{H^s\times H^{s-1}}.$$

Furthermore, the inhomogeneous frequency localized Strichartz estimate, when summed over $k\in\mathbb{Z}$, furnishes

$$\begin{aligned}\sum_k \Big\|P_k\partial_t\int_0^t U(t-s)\big(u_t^2\big)\,ds\Big\|_{L_t^pL_x^\infty([-T,T]\times\mathbb{R}^3)}&\leq C\cdot\sum_k 2^{s_1k}\cdot \big\|P_k(u_t^2)\big\|_{L_t^1L_x^2([-T,T]\times\mathbb{R}^3)}\\&\leq C_1\cdot T^{\frac{1}{p'}}\cdot \big\|u\big\|_X^2.\end{aligned}$$

We complete bounding the contribution of the inhomogeneous term by observing that

$$\big(\sum_k \Big\|\nabla_{t,x}\int_0^t U(t-s)P_k\big(u_t^2\big)\,ds\Big\|^2_{L_t^\infty H^{s-1}([-T,T]\times\mathbb{R}^3)}\big)^{\frac12}\leq C_7\big(\sum_k \big\|P_k(u_t^2)\big\|^2_{L_t^1H_x^{s-1}([-T,T]\times\mathbb{R}^3)}\big)^{\frac12}\leq C_8\cdot T^{\frac{1}{p'}}\cdot \big\|u\big\|_X^2,$$

where we have taken advantage of the energy inequality and the preceding lemma. Combining the previous bounds, we infer that

$$\big\|Tu\big\|_X\leq C_6\cdot \big\|(f,g)\big\|_{H^s\times H^{s-1}}+C_8\cdot T^{\frac{1}{p'}}\cdot \big\|u\big\|_X^2$$

and one deduces a similar bound for differences $Tu-Tv$. Picking T small enough, the theorem is a consequence of the Banach fixed point theorem. □

Remark 2.3. One can show in analogy to lecture 5 that if the data are of any higher regularity $H^{s'}, s'>2$, then this higher regularity is preserved for the solution just constructed, on its interval of existence.

3. Sharpness of the result

It is remarkable that the previous result is essentially sharp. In fact, the following result was proved by H. Lindblad in 1993:

Theorem 3.1. *(Lindblad, 1993) The problem* (1.1) *is ill-posed in* $H^s(\mathbb{R}^3)\times H^{s-1}(\mathbb{R}^3)$ *for any* $s<2$. *Specifically, there is no*

$$T=T(\big\|(f,g)\big\|_{H^s(\mathbb{R}^3)\times H^{s-1}(\mathbb{R}^3)}>0$$

such that for any data pair $(f,g)\in H^s(\mathbb{R}^3)\times H^{s-1}(\mathbb{R}^3)$ *there is a solution*

$$u\in C^0([-T,T],H^s(\mathbb{R}^3))\cap C^1([-T,T],H^s(\mathbb{R}^3)).$$

In fact, for any $s < 2$ there is data $(f, g) \in H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ such that there is no solution on any interval $[-T', T'] \times \mathbb{R}^3$, i. e. the solution collapses instantaneously.

Proof. The idea is to combine the existence of a very special blow up solution for (1.1), namely an *ODE type blow up* which arises when one considers space-indepdendent solutions, with the symmetries acting on the equation, and more specifically a combination of Lorentz transforms and scaling.

Step 1: *A special blow up solution.* Note that setting $u(t, x) = \log t$, we indeed have

$$-u_{tt} = u_t^2.$$

Step 2: *Symmetries acting on solutions.* Here we take advantage of a very special feature of wave equations, namely their *invariance under Lorentz transformations.* Consider the transformation $\mathcal{L}$ defined in terms of

$$\big(\mathcal{L}u\big)(t, x_1, x_2, x_3) = u\big(\frac{t - vx_1}{\sqrt{1-v^2}}, \frac{x_1 - tv}{\sqrt{1-v^2}}, x_2, x_3\big),$$

where $|v| < 1$ is a fixed real number. Then one checks directly that if

$$\Box u = F,$$

then we have

$$\Box\big(\mathcal{L}u\big) = \mathcal{L}F.$$

Now if we apply this Lorentz transform to the special solution $u(t, x) = \log t$, we do not obtain a solution of the original equation, but we have

$$\Box\big(\mathcal{L}\log t\big) = (1 - v^2) \cdot \big[\partial_t\big(\mathcal{L}\log t\big)\big]^2.$$

To remedy this, we instead consider

$$u_v(t, x) := (1 - v^2) \cdot \mathcal{L}\log t = (1 - v^2) \cdot \log\Big(\frac{t - vx_1}{\sqrt{1-v^2}}\Big),$$

which now indeed satisfies (1.1). Obviously this solution becomes singular when $t = x_1 = 0$. We shall let $v \to 1$ and show that the $H^{2-\varepsilon}$-norm of the data at time $t = 1$ converges to zero. A simple re-scaling of the data then shows that we cannot ensure a uniform interval of existence of solutions and preservation of higher regularity provided we only control the $H^{2-\varepsilon}$-norm of the data, see also Remark 2.3. For later purposes, let us set

$$\widetilde{u}_v(t, x) := \psi(x_1) \cdot u_v(t, x),$$

where $\psi \in C_0^\infty(\mathbb{R})$ equals 1 for $|x_1| \leq 2$ and vanishes for $|x_1| \geq 4$, say. The preceding is then no longer a solution of the equation, but shall be handy in the next step, and of course co-incides with u_v for $|x_1| \leq 2$.

Step 3: *Precise construction of the data at time $t = 1$ and verification it is in $H^s, s < 2$.* We need to ensure that $t - vx_1$ does not vanish at time $t = 1$ on the support of the data, since we want the data to be smooth. Thus we shall consider the data pair[1]

$$\phi_v[1] := \big(\widetilde{u}_v(1, x) \cdot \chi(\frac{3(x_1 - 1)}{v^{-1} - 1}) \cdot \chi\big(\frac{x_2^2 + x_3^2}{1 - v^2 x_{1,+}^2}\big), \widetilde{u}_{v,t}(1, x) \cdot \chi(\frac{3(x_1 - 1)}{v^{-1} - 1}) \cdot \chi\big(\frac{x_2^2 + x_3^2}{1 - v^2 x_{1,+}^2}\big)\big),$$

where $\chi \in C^\infty(\mathbb{R})$ is chosen such that $\chi|_{(-\infty,1]} = 1$ and $\mathrm{supp}(\chi) \subset (-\infty, 2)$. Then observe that

$$x_1 \leq \frac{2}{3} \cdot (v^{-1} - 1) + 1 < v^{-1}$$

on the support, which means that $1 - x_1 v \neq 0$ there, and the data are indeed C^∞. Moreover, if $|x| \leq 1$, we have that

$$x_1 - 1 \leq 0,\ x_2^2 + x_3^2 \leq 1 - x_1^2 \leq 1 - v^2 x_{1,+}^2$$

and so

$$\phi_v[1] = u_v[1]$$

[1]We use the notation $x_+ = \max\{x, 0\}$.

on the set $\{|x| \leq 1\}$. The Huyghen's principle implies that the backward solution corresponding to data $\phi_v[1]$ will coincide with $u_v(t,x)$ on the forward light cone $|x| \leq t$, $0 < t < 1$, and in fact even a slightly larger forward light cone. We finally claim

Lemma 3.2. *We have the bounds*

$$\big\|\phi_v[1]\big\|_{H^2(\mathbb{R}^3)\times H^1(\mathbb{R}^3)} \leq C \cdot \big|\log(1-v)\big|,\ \big\|\phi_v[1]\big\|_{H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)} \leq C_1 \cdot (1-v) \cdot \big|\log(1-v)\big|.$$

Proof. (sketch) The idea is that when x_1 is fixed, the coordinates (x_2, x_3) range over a disc of radius $\sqrt{2(1-x_{1,+}^2 v^2)} \leq C \cdot \sqrt{1-vx_1}$. The area of this is bounded by $C_1 \cdot (1-x_1 v)$. Then one directly estimates the integrals of the square of $\partial_{x_1}^2 \phi_v$ etc. For example, when both x_1-derivatives fall on the factor

$$\log\Big(\frac{t-vx_1}{\sqrt{1-v^2}}\Big)|_{t=1},$$

one obtains

$$\frac{1}{(1-x_1 v)^2},$$

and one is led to estimate the integral

$$(1-v^2)^2 \cdot \int \frac{1}{(1-x_1 v)^4} \cdot \chi^2\Big(\frac{3(x_1-1)}{v^{-1}-1}\Big) \cdot C(1-vx_1)\,dx_1 \leq C_2.$$

If none of the ∂_{x_1}-derivatives fall on the logarithmic factor, then one loses an additional $\big|\log(1-v)\big|$. □

Using the preceding lemma and interpolation, we see that for $\varepsilon > 0$ we have

$$\big\|\phi_v[1]\big\|_{H^{2-\varepsilon}(\mathbb{R}^3)\times H^{1-\varepsilon}(\mathbb{R}^3)} \leq C\big|\log(1-v)\big| \cdot (1-v)^{\frac{\varepsilon}{2}},$$

whence

$$\lim_{v\to 1} \big\|\phi_v[1]\big\|_{H^{2-\varepsilon}(\mathbb{R}^3)\times H^{1-\varepsilon}(\mathbb{R}^3)} = 0.$$

Step 4: *Rescaling of the data.* Given a solution $u(t,x)$ of (1.1), we can re-scale it to

$$u_\lambda(t,x) = u(\lambda t, \lambda x)$$

which is still a solution of (1.1). Moreover, we check that

$$\big\|u_\lambda[0]\big\|_{H^{2-\varepsilon}(\mathbb{R}^3)\times H^{1-\varepsilon}(\mathbb{R}^3)} \leq \lambda^{\frac{1}{2}-\varepsilon} \cdot \big\|u[0]\big\|_{H^{2-\varepsilon}(\mathbb{R}^3)\times H^{1-\varepsilon}(\mathbb{R}^3)}$$

provided $\lambda \geq 1$ and $\varepsilon < \frac{1}{2}$.
Given $\varepsilon > 0$, pick a sequence $\{v_n\}_{n\geq 1} \subset (0,1)$ with $\lim_{n\to\infty} v_n = 1$, and further pick $\lambda_n > 0$ such that setting

$$\psi_{v_n,\lambda_n}(t,x) := \phi_{v_n}(1-\lambda_n t, \lambda_n x),$$

we have

$$\big\|\psi_{v_n,\lambda_n}[0]\big\|_{H^{2-\varepsilon}(\mathbb{R}^3)\times H^{1-\varepsilon}(\mathbb{R}^3)} = 1.$$

Necessarily we have $\lambda_n \to +\infty$, and the solutions ψ_{v_n,λ_n} only exist on $[0, \lambda_n^{-1}) \times \mathbb{R}^3$, in spite of the fact that the initial data are uniformly bounded in $H^{2-\varepsilon}(\mathbb{R}^3) \times H^{1-\varepsilon}(\mathbb{R}^3)$. One can construct a solution breaking down instantaneously by combining an infinite sequence of such solutions which are spaced apart and using Huyghen's principle. □

4. A similar example with yet better local well-posedness properties

Here we show that some superficially structurally similar equations have much better well-posedness properties. Again restricting to $\mathbb{R}^{1+3}$, consider the following

$$-u_{tt} + \triangle u = u_t^2 - |\nabla_x u|^2. \tag{4.1}$$

Note that this model also admits the solution $u(t,x) = \log t$, but by contrast to (1.1), the preceding model is actually *invariant under Lorentz transforms*, which means we cannot apply the same construction to it as for (1.1). In fact, we have the much stronger

Theorem 4.1. *(Nirenberg) The model* (4.1) *on* $\mathbb{R}^{1+3}$ *is strongly locally well-posed for data* $(f, g) \in H^{\frac{3}{2}+\varepsilon} \times H^{\frac{1}{2}+\varepsilon}$ *for any* $\varepsilon > 0$*. Thus we gain* $1/2$*-derivative compared to* (1.1)*.*

Proof. (sketch) This is based on the fact that a nice trick transformation transforms this model into the *free wave equation.* Specifically, considering

$$\phi = e^u,$$

we immediately verify that

$$\Box\phi = 0.$$

Now if $u[0] \in H^{\frac{3}{2}+\varepsilon}(\mathbb{R}^3) \times H^{\frac{1}{2}+\varepsilon}(\mathbb{R}^3)$, then $\nabla_{t,x}\phi(0) \in H^{\frac{1}{2}+\varepsilon}(\mathbb{R}^3)$ by Taylor expanding the exponential and using the results of lecture 5 and the Sobolev embedding $H^{\frac{3}{2}+}(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$. Furthermore, we have that

$$\phi(0, x) \geq e^{-C\cdot\left\|u(0,\cdot)\right\|_{H^{\frac{3}{2}+\varepsilon}}} > 0\,\forall x \in \mathbb{R}^3.$$

By global well-posedness of the wave equation in $H^s(\mathbb{R}^3)$, we see that

$$\nabla_{t,x}\phi(t) \in H^{\frac{1}{2}+\varepsilon}(\mathbb{R}^3)\,\forall t \in \mathbb{R},$$

and by using a suitable interpolation argument, one shows that for $0 \leq t < 1$ and $0 < \varepsilon < 1$,

$$\left|\phi(t, x) - \phi(0, x)\right| \leq C_\varepsilon \cdot |t|^{\frac{\varepsilon}{2}} \cdot \left\|(\nabla_x\phi, \phi_t)\right\|_{H^{\frac{1}{2}+\varepsilon}\times H^{\frac{1}{2}+\varepsilon}} \leq C_\varepsilon \cdot |t|^{\frac{\varepsilon}{2}} \cdot e^{C\cdot\left\|u(0,\cdot)\right\|_{H^{\frac{3}{2}+\varepsilon}}}.$$

In fact, to derive this estimate, it suffices to split

$$\left|\phi(t, x) - \phi(0, x)\right| \leq \left|P_{\geq K}\phi(t, x) - P_{\geq K}\phi(0, x)\right| + \left|P_{<K}\phi(t, x) - P_{<K}\phi(0, x)\right|$$

where we set $K := |t|^{-\frac{\varepsilon}{2}}$, and to estimate the two terms on the right separately. It follows that provided we pick

$$T < \left(C_1^{-1} \cdot e^{-C_2\cdot\left\|(u(0,\cdot),u_t(0,\cdot))\right\|_{H^{\frac{3}{2}+\varepsilon}\times H^{\frac{1}{2}+\varepsilon}}}\right)^{\varepsilon^{-1}},$$

we can ensure that

$$\phi(t, x) > c(\left\|(u(0, \cdot), u_t(0, \cdot))\right\|_{H^{\frac{3}{2}+\varepsilon}\times H^{\frac{1}{2}+\varepsilon}}) > 0,\, t \in [-T, T].$$

It then follows that

$$u(t, x) = \log\phi(t, x),\, t \in [-T, T],$$

is also in $H^{\frac{3}{2}+\varepsilon}$, with derivatives in $H^{\frac{1}{2}+\varepsilon}$. For this one really needs a further estimate in the spirit of lecture 5, namely H^s-estimates for compositions of H^s functions with smooth functions.

$\square$