STRICHARTZ ESTIMATES

Our goal now is to deduce certain estimates for the linear Schrodinger and wave equation which will allow us to significantly improve the local well-posedness established in lecture 5, and in fact even derive global well-posedness results in a more general setting. These estimates, named after R. Strichartz who established a special case, will be deduced separately for the Schrodinger case, where we dispose over better dispersive estimates at this point, and then for the wave case. As we had already hinted at earlier in the course, the essence behind these estimates is to use a kind of interpolation between the dispersive amplitude decay and the energy conservation.

1. The Schrodinger case

Recall that the homogeneous Schrodinger propagator is given in the form

$$Tf(t,x) = (2\pi)^{-n} \cdot \int_{\mathbb{R}^n} e^{-it|\xi|^2 + ix\cdot\xi} \cdot \widehat{f}(\xi)\, d\xi.$$

Taking advantage of Lemma 2.1 from lecture1.pdf, we easily infer

Lemma 1.1. *We have the estimate*

$$\big\|Tf(t,\cdot)\big\|_{L_x^\infty(\mathbb{R}^n)} \leq (4\pi|t|)^{-\frac{n}{2}} \cdot \big\|f\big\|_{L^1(\mathbb{R}^n)}.$$

Proof. By density of $\mathcal{S}(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$, we may assume $f \in \mathcal{S}(\mathbb{R}^n)$. Then we can easily write

$$\begin{aligned}
&(2\pi)^{-n} \cdot \int_{\mathbb{R}^n} e^{-it|\xi|^2 + ix\cdot\xi} \cdot \widehat{f}(\xi)\, d\xi \\
&= \lim_{\varepsilon\downarrow 0} (2\pi)^{-n} \cdot \int_{\mathbb{R}^n} e^{-it|\xi|^2 + ix\cdot\xi - \varepsilon|\xi|^2} \cdot \widehat{f}(\xi)\, d\xi \\
&= \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n} \mathcal{F}_{\mathbb{R}^n}^{-1}\big(e^{-it|\xi|^2 - \varepsilon|\xi|^2}\big)(y) \cdot f(x-y)\, dy,
\end{aligned}$$

where $\mathcal{F}_{\mathbb{R}^n}$ denotes the Fourier transform on $\mathbb{R}^n$. Since we have

$$e^{-it|\xi|^2 - \varepsilon|\xi|^2} = \prod_{j=1}^n e^{-it\xi_j^2 - \varepsilon\xi_j^2},$$

we have

$$\mathcal{F}_{\mathbb{R}^n}^{-1}\big(e^{-it|\xi|^2 - \varepsilon|\xi|^2}\big)(y) = \prod_{j=1}^n \mathcal{F}_{\mathbb{R}}^{-1}\big(e^{-it\xi_j^2 - \varepsilon\xi_j^2}\big)(y_j),$$

and lemma 2.1 from lecture1.pdf implies that

$$\lim_{\varepsilon\downarrow 0} \mathcal{F}_{\mathbb{R}^n}^{-1}\big(e^{-it|\xi|^2 - \varepsilon|\xi|^2}\big)(y) = \big(4\pi i t\big)^{-\frac{n}{2}} \cdot e^{i\frac{|y|^2}{4t}},$$

which implies the lemma.

□

We next recall the L^2-conservation, in the form

$$\big\|Tf(t,\cdot)\big\|_{L_x^2(\mathbb{R}^n)} = \big\|f\big\|_{L^2(\mathbb{R}^n)},\ t \in \mathbb{R}.$$

We now use a well-known *interpolation result*, namely the *Riesz-Thorin theorem*, which we state here without proof:

Theorem 1.2. *(Riesz-Thorin) Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, and assume (X, μ) is a measure space. Assume that T is a linear operator defined on all simple functions on X and taking values in the measurable functions on X, which satisfies*

$$\|T(f)\|_{L^{q_0}(X)} \leq A_0\|f\|_{L^{p_0}(X)},\ \|T(f)\|_{L^{q_1}(X)} \leq A_1\|f\|_{L^{p_1}(X)}.$$

for all simple functions f. Then for any $p \in [p_0, p_1]$, with $\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$, $\theta \in [0, 1]$, we have

$$\|T(f)\|_{L^q} \leq A_0^\theta A_1^{1-\theta}\|f\|_{L^p},$$

where q is defined via $\frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}$.

We use this with $p_0 = 1, p_1 = 2$, as well as $q_0 = \infty, q_1 = 2$, and note that then p, q as determined by the preceding theorem (with variable $\theta \in [0, 1]$) will be Holder dual:

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Using furthermore $A_0 = C \cdot |t|^{-\frac{n}{2}}$, $A_1 = 1$, we find

Proposition 1.3. *For any $p \in [2, \infty]$, letting $p' \in [1, 2]$ denote the Holder dual Lebesgue exponent, we have*

$$\big\|Tf(t, \cdot)\big\|_{L_x^p(\mathbb{R}^n)} \leq C \cdot |t|^{-n\cdot(\frac{1}{2}-\frac{1}{p})} \cdot \big\|f\big\|_{L^{p'}(\mathbb{R}^n)}.$$

Given that there is some temporal decay in the preceding estimate for $p \in (2, \infty]$, it is natural to *integrate this bound over time* in a suitably weighted fashion. This is essentially the intuition behind the following important

Theorem 1.4. *Let $q \geq 2, \infty \geq p > 2$ and furthermore assume the sharp Strichartz admissibility condition*

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}. \tag{1.1}$$

Then we have the bound

$$\big\|Tf\big\|_{L_t^p L_x^q(\mathbb{R}^{1+n})} \leq C_{n,p,q} \cdot \big\|f\big\|_{L_x^2(\mathbb{R}^n)}. \tag{1.2}$$

Remark 1.5. The condition (1.1) is in fact dictated by simple scaling considerations, replacing

$$Tf(t, x) =: u(t, x)$$

by

$$u_\lambda(t, x) := u(\lambda^2 t, \lambda x),\ \lambda > 0$$

and $f(x)$ by $f_\lambda(x) := f(\lambda x)$. Then we still have

$$(i\partial_t + \triangle)u_\lambda = 0,\ u_\lambda(0, \cdot) = f_\lambda(\cdot),$$

and the left and right hand sides of (1.2) scale the same way precisely when (1.1) is satisfied.

The proof of Theorem 1.4 relies on a reformulation, taking advantage of the *dual operator* T^* which is defined as follows: for any $g \in \mathcal{S}(\mathbb{R}^{1+n})$ and $f \in \mathcal{S}(\mathbb{R}^n)$ we require

$$\langle Tf, g\rangle_{L^2(\mathbb{R}^{1+n})} = \langle f, T^* g\rangle_{L^2(\mathbb{R}^n)},$$

which implies that

$$T^* g(x) = (2\pi)^{-n} \int_{\mathbb{R}^{1+n}} e^{it|\xi|^2 + ix\cdot\xi} \cdot \widehat{g}(t, \xi)\, d\xi dt,$$

In the sequel we shall assume $2 \leq p < \infty$, $q > 2$, since the case $p = \infty, q = 2$ is already known to us. Then we have

Lemma 1.6. *The operator T extends to a bounded operator between $L_x^2(\mathbb{R}^n)$ and $L_t^p L_x^q(\mathbb{R}^n)$ iff T^* extends to a bounded operator between $L_t^{p'} L_x^{q'}(\mathbb{R}^{1+n})$ and $L_x^2(\mathbb{R}^n)$. Moreover, this is the case iff the composition $T \circ T^*$ extends as a bounded operator between $L_t^{p'} L_x^{q'}(\mathbb{R}^{1+n})$ and $L_t^p L_x^q(\mathbb{R}^{1+n})$.*

Proof. Note that for $f, g \in \mathcal{S}(\mathbb{R}^{1+n})$, we have

$$\langle f, T\circ T^* g\rangle_{L^2(\mathbb{R}^{1+n})} = \langle T^* f, T^* g\rangle_{L^2(\mathbb{R}^n)}$$

and so

$$\big\|T\circ T^* g\big\|_{L_t^p L_x^q} = \sup_{\|f\|_{L_t^{p'} L_x^{q'}}=1} \langle f, T\circ T^* g\rangle_{L^2(\mathbb{R}^{1+n})} \leq \sup_{\|f\|_{L_t^{p'} L_x^{q'}}=1} \big\|T^* f\big\|_{L_x^2(\mathbb{R}^n)} \cdot \big\|T^* g\big\|_{L_x^2(\mathbb{R}^n)}.$$

which implies the boundedness of $T\circ T^*$ as operator between $L_t^{p'} L_x^{q'}(\mathbb{R}^{1+n})$ and $L_t^p L_x^q(\mathbb{R}^{1+n})$ provided that of T^* is known.

Conversely, if we assume $T\circ T^*$ is bounded, we have

$$\big\|T^* f\big\|_{L_x^2(\mathbb{R}^n)}^2 = \big|\langle f, T\circ T^* f\rangle_{L^2(\mathbb{R}^{1+n})}\big| \leq \big\|f\big\|_{L_t^{p'} L_x^{q'}} \cdot \big\|T\circ T^* f\big\|_{L_t^p L_x^q}$$

which implies the boundedness of T^*.

Observe that the preceding argument furnishes the norm equality

$$\big\|T_*\big\|^2 = \big\|T\circ T^*\big\|.$$

The first part of the lemma is standard. □

Thanks to the preceding lemma, the proof of the Theorem will follow once we can prove an a priori bound for the operator

$$T\circ T^* g(t,\cdot) = (2\pi)^{-n}\int_{\mathbb{R}^{1+n}} e^{i(s-t)|\xi|^2 + ix\cdot\xi}\cdot \widehat{g}(s,\xi)\, d\xi ds. \tag{1.3}$$

when acting between the spaces $L_t^{p'} L_x^{q'}(\mathbb{R}^{1+n})$ and $L_t^p L_x^q(\mathbb{R}^{1+n})$ with p, q satisfying the conditions of the theorem.

This shall be a direct consequence of Proposition 1.3 in conjunction with the following version of the Sobolev inequality in disguise, which is called *Hardy-Littlewood-Sobolev inequality*:

Proposition 1.7. *Let $0<\alpha<1$ and define for $f\in\mathcal{S}(\mathbb{R})$*

$$Uf(t) = \int_{\mathbb{R}} |t-s|^{-\alpha} f(s)\, ds$$

Then we have

$$\big\|Uf\big\|_{L_t^p(\mathbb{R})} \leq C_{\alpha,p}\cdot \big\|f\big\|_{L_s^q(\mathbb{R})}$$

provided we have $q>1$ and

$$\frac{1}{q} - \frac{1}{p} = 1-\alpha.$$

We provide a quick proof for the case $1<q\leq 2$, $\infty>p\geq 2$, which will be the one relevant to us, based on the fundamental theorem of Littlewood-Paley theory. First we observe

Lemma 1.8. *Setting for $0<\alpha<1$*

$$\big(\widehat{|t|^{-\alpha}}\big)(\xi) = \lim_{M\to+\infty}\int_{\mathbb{R}} \chi_{|t|\leq M}\cdot \frac{e^{-it\xi}}{|t|^\alpha}\, dt,$$

we have

$$\big(\widehat{|t|^{-\alpha}}\big)(\xi) = c_\alpha\cdot|\xi|^{-1+\alpha}.$$

for a suitable constant $c_\alpha\in\mathbb{R}$.

Proof. We have for $\xi\neq 0$

$$\begin{aligned}
\widehat{|t|^{-\alpha}}(\xi) &= \lim_{M\to+\infty}\int_{\mathbb{R}} \chi_{|t|\leq M}\cdot\frac{e^{-it\xi}}{|t|^\alpha}\, dt \\
&= |\xi|^{-1+\alpha}\cdot \lim_{M\to+\infty}\int_{\mathbb{R}} \chi_{|y|\leq |\xi| M}\cdot \frac{e^{-iy}}{|y|^\alpha}\, dy \\
&= c_\alpha\cdot|\xi|^{-1+\alpha}
\end{aligned}$$

where we set

$$c_\alpha := \int_{\mathbb{R}} \frac{e^{-iy}}{|y|^\alpha}\,dy$$

□

Proof. (Prop. 1.7, $q \leq 2$, $\infty > p \geq 2$) To begin with, recall from Cor. 3.3 from lecture3.pdf that

$$\big\|Uf\big\|_{L^p(\mathbb{R})} \leq C \cdot \big(\sum \|P_l(Uf)\|_{L^p(\mathbb{R})}^2\big)^{\frac{1}{2}}.$$

According to the preceding lemma, we have

$$\widehat{P_l(Uf)} = c_\alpha \frac{\psi_l(\xi)}{|\xi|^{1-\alpha}} \cdot \widehat{f}(\xi),$$

where ψ_l is as in the definition of the multiplier P_l in lecture3.pdf. Then we state

Lemma 1.9. *If q, p satisfy $\frac{1}{q} - \frac{1}{p} = 1 - \alpha$, $\alpha \in (0, 1)$, as well as $q \geq 1$, then we have for any $l \in \mathbb{Z}$*

$$\Big\|\mathcal{F}^{-1}\big(\frac{\psi_l(\xi)}{|\xi|^{1-\alpha}} \cdot \widehat{f}(\xi)\big)\Big\|_{L^p(\mathbb{R})} \leq C \cdot \big\|P_l f\big\|_{L^q(\mathbb{R})},$$

where $C = C(\alpha, p)$, and where $\mathcal{F}^{-1}$ is the inverse Fourier transform.

Proof. The starting point is *Young's inequality*, which we state as follows: letting

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y) \cdot g(y)\,dy,$$

then we have

$$\big\|f * g\big\|_{L^r(\mathbb{R}^n)} \leq \big\|f\big\|_{L^p(\mathbb{R}^n)} \cdot \big\|g\big\|_{L^q(\mathbb{R}^n)}$$

provided $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. The proofs follows from the cases $r = \infty$ and $r = p$ (Minkowski's inequality) by means of interpolation (Riesz-Thorin). Returning to the proof of the lemma, observe that

$$\mathcal{F}^{-1}\big(\frac{\psi_l(\xi)}{|\xi|^{1-\alpha}}\big)(y) = 2^{\alpha l} \cdot \zeta_{l,\alpha}(y),$$

where we have the bound

$$\big|\zeta_{l,\alpha}(y)\big| \leq C_{N,\alpha,n} \cdot (1 + 2^l y)^{-N}$$

for any $N > 0$. Hence

$$\big\|2^{\alpha l} \cdot \zeta_{l,\alpha}\big\|_{L^{\alpha^{-1}}(\mathbb{R}^n)} \leq D_{\alpha,n},$$

and the lemma follows from Young's inequality. □

Continuing with the proof of the proposition, we now infer

$$\big\|Uf\big\|_{L^p(\mathbb{R})} \leq C \cdot \big(\sum \|P_l f\|_{L^q(\mathbb{R})}^2\big)^{\frac{1}{2}} \tag{1.4}$$

To conclude we need one more

Lemma 1.10. *Assume that $1 < q \leq 2$. Then there is a universal constant $C_{q,n}$ such that for each $g \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$\big(\sum_{l\in\mathbb{Z}} \big\|P_l g\big\|_{L^q(\mathbb{R}^n)}^2\big)^{\frac{1}{2}} \leq C_{q,n} \cdot \big\|g\big\|_{L^q(\mathbb{R}^n)}.$$

Proof. This assertion is 'dual' to the one of Cor. 3.3 in lecture3.pdf. Assume as we may that g is real valued. Observe that (with $\frac{1}{p}+\frac{1}{q}=1$)

$$\begin{aligned}\big(\sum_{l\in\mathbb{Z}}\big\|P_l g\big\|_{L^q(\mathbb{R}^n)}^2\big)^{\frac{1}{2}} &= \sup_{\sum_l \|f_l\|_{L^p(\mathbb{R}^n)}^2\leq 1}\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n} f_l\cdot P_l g\,dx\\ &= \sup_{\sum_l \|f_l\|_{L^p(\mathbb{R}^n)}^2\leq 1}\int_{\mathbb{R}^n}\sum_{l\in\mathbb{Z}} P_l f_l\cdot g\,dx\\ &\leq \sup_{\sum_l \|f_l\|_{L^p(\mathbb{R}^n)}^2\leq 1}\big\|\sum_{l\in\mathbb{Z}} P_l f_l\big\|_{L^p(\mathbb{R}^n)}\cdot\big\|g\big\|_{L^q(\mathbb{R}^n)},\end{aligned}$$

where we have used Holder's inequality in the last step. Setting $f:=\sum_{l\in\mathbb{Z}} P_l f_l$, we have

$$\begin{aligned}\big\|f\big\|_{L^p(\mathbb{R}^n)} &\leq C\cdot\big(\sum_l\big\|P_l f\big\|_{L^p(\mathbb{R}^n)}^2\big)^{\frac{1}{2}}\\ &\leq \widetilde{C}\cdot\big(\sum_l\big\|f_l\big\|_{L^p(\mathbb{R}^n)}^2\big)^{\frac{1}{2}},\end{aligned}$$

where for the first inequality we have taken advantage of Cor. 3. 3. in lecture3.p df and the fact that $p\geq 2$. The lemma follows.

□

The proof of the proposition (for $q\leq 2, p\geq 2$) now follows from (1.4) and the preceding lemma. □

We have all the tools to now complete the proof of Theorem 1.4:

Proof. (Theorem 1.4) Consider the composition

$$T\circ T^*(g)$$

given in (1.3). From Prop. 1.3 we know that

$$\big\|T\circ T^*(g)(t,\cdot)\big\|_{L_x^q(\mathbb{R}^n)}\leq C\cdot\int_{\mathbb{R}}|t-s|^{-\alpha}\cdot\big\|g(s\cdot)\big\|_{L_x^{q'}(\mathbb{R}^n)}\,ds,$$

provided we set

$$\alpha = n\cdot(\frac{1}{2}-\frac{1}{q}).$$

According to the admissibility conditions in the theorem, we have

$$0<\alpha=\frac{2}{p}<1,$$

and so we can apply Prop. 1.7. Observing that

$$\frac{1}{p'}-\frac{1}{p}=1-\frac{2}{p}=1-\alpha,$$

we find

$$\Big\|T\circ T^*(g)\Big\|_{L_t^pL_x^q(\mathbb{R}^{1+n})}\leq C_{p,q,n}\cdot\big\|g\big\|_{L_t^{p'}L_x^{q'}(\mathbb{R}^{1+n})}.$$

The theorem is then a consequence of Lemma 1.6. □

Remark 1.11. The inequality is also true for $p=2, q=\frac{2n}{n-2}$, $n\geq 3$. This so-called endpoint case is due to Keel-Tao.

We also mention the following result, based on a technical lemma called the 'Christ-Kiselev lemma', and which allows one to infer boundedness of the Duhamel propagator:

Theorem 1.12. *Assume that (p,q) and $(\widetilde{p},\widetilde{q})$ are Strichartz admissible in the sense of the preceding theorem. Then we have the bound*

$$\Big\|\int_0^t S(t-s)F(s,\cdot)\,ds\Big\|_{L_t^pL_x^q(\mathbb{R}^{1+n})}\leq C\cdot\big\|F\big\|_{L_t^{\widetilde{p}'}L_x^{\widetilde{q}'}(\mathbb{R}^{1+n})}$$

2. The wave case

Here the situation is a bit more complicated as we cannot reduce the derivation of the required general dispersive amplitude decay bound to the one dimensional case as was the case for the Schrodinger equation. Instead, we shall have to derive the necessary bound from first principles, and more specifically from basic harmonic analysis techniques. From now on we shall assume $n\geq 2$ throughout since else there is no dispersive decay. Recall that the solution of the linear wave equation

$$\Box u=(-\partial_{tt}+\triangle_{\mathbb{R}^n})u=0,\;u[0]=(f,g)$$

is given by means of the Fourier formula

$$\begin{aligned}u(t,x)&=(2\pi)^{-n}\cdot\int_{\mathbb{R}^n}e^{i(t|\xi|+x\cdot\xi)}\cdot\frac{1}{2}[\widehat{f}(\xi)+\frac{1}{i|\xi|}\widehat{g}(\xi)]\,d\xi\\&+(2\pi)^{-n}\cdot\int_{\mathbb{R}^n}e^{i(-t|\xi|+x\cdot\xi)}\cdot\frac{1}{2}[\widehat{f}(\xi)-\frac{1}{i|\xi|}\widehat{g}(\xi)]\,d\xi\end{aligned}\tag{2.1}$$

We shall for now consider the *frequency localized waves*

$$\begin{aligned}u_0(t,x)&=(2\pi)^{-n}\cdot\int_{\mathbb{R}^n}\psi(\xi)e^{i(t|\xi|+x\cdot\xi)}\cdot\frac{1}{2}[\widehat{f}(\xi)+\frac{1}{i|\xi|}\widehat{g}(\xi)]\,d\xi\\&+(2\pi)^{-n}\cdot\int_{\mathbb{R}^n}\psi(\xi)e^{i(-t|\xi|+x\cdot\xi)}\cdot\frac{1}{2}[\widehat{f}(\xi)-\frac{1}{i|\xi|}\widehat{g}(\xi)]\,d\xi\end{aligned}$$

where $\psi\in C_0^\infty(\mathbb{R}^n)$ is as in the definition of the Littlewood-Paley cutoffs and hence supported at $|\xi|\sim 1$. The desired dispersive decay for this special case will follow from

Proposition 2.1. *We have the estimate*

$$\Big\|\int_{\mathbb{R}^n}\psi(\xi)e^{i(t|\xi|+x\cdot\xi)}\cdot\widehat{f}(\xi)\,d\xi\Big\|_{L_x^\infty(\mathbb{R}^n)}\lesssim(1+|t|)^{-\frac{n-1}{2}}\cdot\big\|f\big\|_{L^1(\mathbb{R}^n)}.$$

The proof of this result shall be based on a *stationary phase argument*, and more specifically Van der Corput's lemma. To begin with, shall shall require the following basic fact due to M. Morse:

Lemma 2.2. *Let $f\in C^\infty(\mathbb{R}^n)$ and assume that $x_*\in\mathbb{R}^n$ is a critical point, i. e. $\nabla f(x_*)=0$. Moreover, assume that this critical point is non-degenerate, in the sense that the Hessian*

$$\big(\frac{\partial^2}{\partial x_i\partial x_j}f(x_*)\big)_{1,\leq i,j\leq n}$$

is invertible. Then there exist neighborhoods U of x_* and V of 0 as well as a diffeomorphism

$$\psi:V\rightarrow U$$

with $\psi(0)=x_*$ and such that (with $c=f(x_*)$)

$$f\circ\psi(y)=c+\sum_{j=1}^n\kappa_jy_j^2,$$

where $\kappa_j=\pm 1$.

We accept this fundamental result without proof. A next ingredient is the following equally important

Lemma 2.3. *Let $\psi \in C_0^\infty(\mathbb{R}^n)$ and consider the integral*

$$F(\lambda) := \int_{\mathbb{R}^n} \psi(x) \cdot e^{i\lambda \sum_{j=1}^n \kappa_j x_j^2}\, dx,\ \kappa_j \in \{\pm 1\}.$$

Then we have the bound

$$\big|F(\lambda)\big| \leq C \cdot \lambda^{-\frac{n}{2}}$$

for $\lambda \gg 1$ and some constant C depending on n, ψ.

Proof. Let $\chi(x) \in C_0^\infty$ a smooth non-negative function which equals 1 for $|x| \leq 1$, and write

$$F(\lambda) = F_1(\lambda) + F_2(\lambda)$$

where

$$F_1(\lambda) := \int_{\mathbb{R}^n} \chi\big(|x| \cdot \sqrt{\lambda}\big)\psi(x) \cdot e^{i\lambda \sum_{j=1}^n \kappa_j x_j^2}\, dx.$$

Then we easily verify that

$$F_1(\lambda) \leq C_1 \cdot \int_{|x|<\sqrt{\lambda}^{-1}} dx \leq C_2 \cdot \lambda^{-\frac{n}{2}}.$$

For F_2, we perform integration by parts, using that

$$e^{i\lambda \sum_{j=1}^n \kappa_j x_j^2} = \big(\frac{1}{(2i\lambda|x|^2)}\sum_l x_l \kappa_l \partial_{x_l}\big)^a e^{i\lambda \sum_{j=1}^n \kappa_j x_j^2}$$

where we set $a = \lfloor \frac{n}{2} \rfloor + 2$. It follows that

$$F_2(\lambda) = \lambda^{-a} \int_{\mathbb{R}^n} e^{i\lambda \sum_{j=1}^n \kappa_j x_j^2} \cdot \big(-\sum_l \partial_{x_l} \kappa_l x_l \frac{1}{(2i|x|^2)}\big)^a \Big[\big(1 - \chi\big(|x| \cdot \sqrt{\lambda}\big)\big)\psi(x)\Big]\, dx.$$

It is directly verified(exercise!) that

$$\big|F_2(\lambda)\big| \leq C_3 \cdot \lambda^{-\frac{n}{2}}.$$

□

Combining the two preceding lemmas, we can infer the following

Lemma 2.4. *Let $\phi \in C^\infty(\mathbb{R}^n)$, real valued, and assume $x_* \in \mathbb{R}^n$ is a non-degenerate critical point. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ and assume that x_* is the only critical point of ϕ on supp(ψ). Then we have the bound*

$$\big|\int_{\mathbb{R}^n} \psi(x) \cdot e^{i\lambda\phi(x)}\, dx\big| \leq C\lambda^{-\frac{n}{2}}$$

provided $\lambda \gg 1$.

Proof. Pick neighbourhoods U of x_* and V of $0 \in \mathbb{R}^n$ such that the conclusion of Lemma 2.2 applies. Let $\psi_1 \in C_0^\infty(\mathbb{R}^n)$ have support in U and write

$$\int_{\mathbb{R}^n} \psi(x) \cdot e^{i\lambda\phi(x)}\, dx = \int_{\mathbb{R}^n} \psi_1(x)\psi(x) \cdot e^{i\lambda\phi(x)}\, dx + \int_{\mathbb{R}^n} (1 - \psi_1(x))\psi(x) \cdot e^{i\lambda\phi(x)}\, dx.$$

Then changing variables for the first integral and interpreting $x = x(y)$ as a function of $y \in V$, we have

$$\int_{\mathbb{R}^n} \psi_1(x)\psi(x) \cdot e^{i\lambda\phi(x)}\, dx = e^{ic\lambda} \int_V \zeta(y) \cdot e^{i\lambda \sum_{j=1}^n \kappa_j y_j^2}\, dy,$$

where we have denoted

$$\zeta(y) := \psi_1(x(y))\psi(x(y)) \cdot \frac{\partial x}{\partial y}.$$

Here $\frac{\partial x}{\partial y}$ is the Jacobian, and so $\zeta \in C_0^\infty(V) \subset C_0^\infty(\mathbb{R}^n)$. We can then apply Lemma 2.3 to infer the desired bound for the first integral on the right above. We conclude by observing the bound

$$\Big|\int_{\mathbb{R}^n} (1-\psi_1(x))\psi(x)\cdot e^{i\lambda\phi(x)}\,dx\Big| \leq C_N\lambda^{-N}$$

for arbitrary $N \geq 1$, which follows by repeated integration by parts (exercise!). □

A consequence of the preceding development is the following key

Proposition 2.5. *Let $d\sigma_\omega$ the surface measure on the sphere $S^{n-1} \subset \mathbb{R}^n$. Then we have*

$$\Big|\int_{S^{n-1}} e^{i\omega\cdot\xi} d\sigma_\omega\Big| \leq C\cdot(1+|\xi|)^{-\frac{n-1}{2}}.$$

Proof. Write

$$\omega = (\omega', \omega_n),\ \omega' \in \mathbb{R}^{n-1},$$

where $|\omega'|^2 + \omega_n^2 = 1$, also write

$$\omega\cdot\xi = \omega'\cdot\xi' + \omega_n\cdot\xi_n.$$

Working in a local coordinate chart (apply partition of unity), we may assume $\omega_n \geq \frac{1}{\sqrt{n}}$, and we reduce to bounding

$$\int_{\mathbb{R}^{n-1}} \psi(\omega')\cdot e^{i(\omega'\cdot\xi' + \omega_n\cdot\xi_n)}\,d\omega'$$

for a suitable function[1] $\psi \in C_0^\infty(B_1(0))$. Compared to the previous lemma, the role of λ is played here by $|\xi|$. Note that if

$$|\xi_n| < \frac{|\xi|}{10\sqrt{n}},$$

then

$$\xi' - \frac{\omega'}{\omega_n}\cdot\xi_n \neq 0,$$

and integration by parts leads to decay to any power in $|\xi|^{-1}$, better than what we need. On the other hand, if $|\xi_n| \geq \frac{|\xi|}{10\sqrt{n}}$, setting

$$\phi(\omega') := \omega'\cdot\frac{\xi'}{|\xi|} + \omega_n\cdot\frac{\xi_n}{|\xi|},$$

we have

$$\nabla_{\omega'}\phi = |\xi|^{-1}(\xi' - \frac{\omega'}{\omega_n}\xi_n),$$

and so there is at most one critical point on the support of the integrand characterized by

$$\frac{\omega'_*}{\omega_{*,n}} := \frac{\xi'}{\xi_n}.$$

Furthermore, one easily checks that the Hessian

$$\nabla^2_{\omega'}\phi(\omega'_*)$$

is non-degenerate(exercise!). The proposition is now a consequence of Lemma 2.4. □

Finally we can give the

Proof. (Prop. 2.1) It suffices to show that

$$\Big|\int_{\mathbb{R}^n} \psi(\xi)e^{i(t|\xi| + x\cdot\xi)}\,d\xi\Big| \leq C\cdot(1+|t|)^{-\frac{n-1}{2}}.$$

For this one distinguishes between two regions:

[1] Here $B_1(0)$ is the ball of radius 1 supported at the origin

(1): $|x| \geq \frac{|t|}{2}$. Using spherical coordinates write

$$\int_{\mathbb{R}^n} \psi(\xi) e^{i(t|\xi|+x\cdot\xi)}\, d\xi = \int_{\mathbb{R}} \int_{S^{n-1}} \psi(|\xi|,\omega) e^{i(t|\xi|+|\xi|x\cdot\omega)}\, |\xi|^{n-1} d\omega d|\xi|$$

and use the preceding proposition to conclude that

$$\Big|\int_{S^{n-1}} \psi(|\xi|,\omega) e^{i(t|\xi|+|\xi|x\cdot\omega)}\, d\omega\Big| \leq C(1+|x|)^{-\frac{n-1}{2}} \leq D(1+|t|)^{-\frac{n-1}{2}}.$$

(2): $|x| < \frac{|t|}{2}$. Here one uses integration by parts(exercise!) with respect to $|\xi|$ to conclude the stronger bound (for any $N \geq 1$)

$$\Big|\int_{\mathbb{R}^n} \psi(\xi) e^{i(t|\xi|+x\cdot\xi)}\, d\xi\Big| \leq C_N \cdot (1+|t|)^{-N}.$$

□

We now have all the preparations to implement the TT^*-argument for the *frequency localized* wave propagator. Proceeding in identical fashion as for the Schrodinger case, we infer the following

Theorem 2.6. *Let $n \geq 2$ and $\infty \geq q \geq 2$, $\infty \geq p > 2$, and assume the sharp wave admissibility condition*

$$\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}.$$

Then denoting

$$T_{\pm} f := \int_{\mathbb{R}^n} \psi(\xi) e^{i(\pm t|\xi|+x\cdot\xi)} \cdot \widehat{f}(\xi)\, d\xi,$$

we have the bound

$$\big\|T_{\pm} f\big\|_{L_t^p L_x^q} \leq C_{p,q,n} \cdot \big\|f\big\|_{L^2(\mathbb{R}^n)}.$$

Remark 2.7. The result is also true in the endpoint case $p = 2$ expect when $n = 3$ (Keel-Tao). Furthermore, the Christ-Kiselev lemma gives a variant for the inhomogeneous Duhamel propagator corresponding to T.

The preceding result only applying to frequency ~ 1 functions, we still need to generalize things to the general setting, which is straightforward with our preparations. To begin with, observe that if $u(t,x)$ is a free wave supported at frequency $\xi \sim \lambda \in 2^{\mathbb{Z}}$, then

$$u_\lambda(t,x) := u(\frac{t}{\lambda}, \frac{x}{\lambda})$$

is a free wave supported at frequency ~ 1. Thus we have the estimate (with $f_\lambda(x) = f(\frac{x}{\lambda})$ the data in the sense of $T_{\pm}$)

$$\big\|u_\lambda\big\|_{L_t^p L_x^q} \leq C \cdot \big\|f_\lambda\big\|_{L^2(\mathbb{R}^n)}.$$

But simple scaling reasons yield

$$\big\|u_\lambda\big\|_{L_t^p L_x^q} = \lambda^{\frac{1}{p}+\frac{n}{q}} \cdot \big\|u\big\|_{L_t^p L_x^q},\ \big\|f_\lambda\big\|_{L^2(\mathbb{R}^n)} = \lambda^{\frac{n}{2}} \cdot \big\|f\big\|_{L^2(\mathbb{R}^n)}.$$

We conclude that for a frequency λ free wave, we have the bound

$$\big\|u\big\|_{L_t^p L_x^q} \leq C \cdot \lambda^{\frac{n}{2}-\frac{1}{p}-\frac{n}{q}} \cdot \big\|f\big\|_{L^2(\mathbb{R}^n)} \sim \big\|f\big\|_{\dot{H}^s(\mathbb{R}^n)}$$

where we set $s = \frac{n}{2} - \frac{1}{p} - \frac{n}{q}$.
For a general free wave, which is not supported at any particular dyadic frequency, introduce the space

$$\big\|u\big\|_{L_t^p \dot{B}^{q,2}(\mathbb{R}^n)} := \Big\|\big(\sum_{k\in\mathbb{Z}} \big\|P_k u\big\|_{L_x^q(\mathbb{R}^n)}^2\big)^{\frac{1}{2}}\Big\|_{L_t^p}$$

Then we can state

Corollary 2.8. *Let (p,q,n) be sharp wave Strichartz admissible. Then we have the estimate*

$$\big\|u\big\|_{L_t^p\dot{B}^{q,2}(\mathbb{R}^n)} \leq C_{p,q,n}\cdot\big\|f\big\|_{\dot{H}^s},\ s=\frac{n}{2}-\frac{1}{p}-\frac{n}{q}.$$

Proof. According to Minkowski's inequality (using $p>2$) and the preceding theorem we have

$$\begin{aligned}\Big\|\big(\sum_{k\in\mathbb{Z}}\big\|P_k u\big\|_{L_x^q(\mathbb{R}^n)}^2\big)^{\frac{1}{2}}\Big\|_{L_t^p} &\leq \big(\sum_{k\in\mathbb{Z}}\big\|P_k u\big\|_{L_t^pL_x^q(\mathbb{R}^{1+n})}^2\big)^{\frac{1}{2}}\\ &\leq C\cdot\big(\sum_{k\in\mathbb{Z}}\big\|P_k f\big\|_{\dot{H}^s(\mathbb{R}^n)}^2\big)^{\frac{1}{2}}\\ &\leq D\cdot\big\|f\big\|_{\dot{H}^s(\mathbb{R}^n)}.\end{aligned}$$

□

We note that the norm $\|\cdot\|_{\dot{B}^{q,2}(\mathbb{R}^n)}$ is a *Besov-type norm.* However, we note that for $2\leq q<\infty$, thanks to the fundamental theorem of Littlewood-Paley theory we have the following inequality

$$\big\|g\big\|_{L^q(\mathbb{R}^n)}\leq C\cdot\big\|g\big\|_{\dot{B}^{q,2}(\mathbb{R}^n)},$$

and so a further consequence of the preceding corollary is the more standard type estimate

$$\Big\|u\Big\|_{L_t^pL_x^q(\mathbb{R}^{1+n})}\leq C\cdot\big\|f\big\|_{\dot{H}^s(\mathbb{R}^n)}$$

for sharp Strichartz admissible (p,q,n) and with $q<\infty$.

Finally, we make one more comment, namely that the sharp Srichartz admissibility condition may be relaxed to the more general admissibility condition:

Corollary 2.9. *Assume that $2<p\leq\infty$, $2\leq q\leq\infty$ and the condition*

$$\frac{2}{p}+\frac{n-1}{q}\leq\frac{n-1}{2}.$$

Then the same conclusion as in the preceding corollary holds.

Proof. Given admissible (p,q) with

$$\frac{2}{p}+\frac{n-1}{q}<\frac{n-1}{2},$$

pick $\widetilde{q}\in[2,\infty)$ such that $(p,\widetilde{q})$ is sharp Strichartz wave admissible. Then use the fact that

$$\big\|u\big\|_{L_t^pL_x^q}\leq C\lambda^{n\cdot(\frac{1}{\widetilde{q}}-\frac{1}{q})}\cdot\big\|u\big\|_{L_t^pL_x^{\widetilde{q}}}$$

for a free wave supported at frequency λ (remaining details left as exercise).

□