TOWARDS GLOBAL WELL-POSEDNESS OF NONLINEAR DISPERSIVE PDE; IMPROVED LOCAL WELL-POSEDNESS IN $H^s(\mathbb{R}^n)$

1. Conserved energy and global well-posedness

In the last lecture we established a local well-posedness result for the class of nonlinear Schrodinger equations

$$i\psi_t + \triangle\psi = \pm|\psi|^{p-1}\psi,\ \psi(0,\cdot) = f \tag{1.1}$$

on $\mathbb{R}^{1+n}$ and data $f \in H^s(\mathbb{R}^n)$, where we assumed (and shall do so below) that p is an odd integer > 1. We had to impose $s > \frac{n}{2}$, and the proof only furnishes the existence of solutions on a time interval I centered around $t = 0$ and whose length depends on $\|f\|_{H^s(\mathbb{R}^n)}$. The proof does not preclude the norm

$$\|\psi(t,\cdot)\|_{H^s(\mathbb{R}^n)}$$

from growing, and so this argument does not furnish a way to prove the existence of a *global-in-time solution*. However, it turns out that if we take advantage of *energy conservation*, then at least in the case $n = 1$ *and the* $+$*-case, which is also called the defocussing case*, we can in fact infer the existence of global solutions for $s \geq 1$. To begin with, we state

Lemma 1.1. *Let* $s \geq 1$ *and also* $s > \frac{n}{2}$. *Then if*

$$\psi \in C^0(I; H^s(\mathbb{R}^n))$$

solves (1.1) *in the Duhamel sense, then the quantity*

$$E := \int_{\mathbb{R}^n} \big(\frac{1}{2}\big|\nabla_x\psi\big|^2 \pm \frac{1}{p+1}|\psi|^{p+1}\big)\,dx$$

is defined for each $t \in I$ *and in fact time independent.*

Remark 1.2. Observe that this quantity is non-negative if we are in the defocussing situation corresponding to the $+$-sign on the right hand side. It is in this situation where we can benefit most from energy conservation.

Proof. To begin with, note that since $\nabla_x\psi \in H^{s-1}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, the integral

$$\int_{\mathbb{R}^n} \frac{1}{2}\big|\nabla_x\psi\big|^2\,dx$$

converges. Since $s > \frac{n}{2}$, Sobolev's embedding implies that

$$\big\|\psi\big\|_{L^\infty(\mathbb{R}^n)} \lesssim \big\|\psi\big\|_{H^s(\mathbb{R}^n)},$$

and so

$$\begin{aligned}\int_{\mathbb{R}^n} \big|\psi\big|^{p+1}\,dx &\leq \big\|\psi\big\|^2_{L^2(\mathbb{R}^n)} \cdot \big\|\psi\big\|^{p-1}_{L^\infty(\mathbb{R}^n)}\\ &\lesssim \big\|\psi\big\|^2_{L^2(\mathbb{R}^n)} \cdot \big\|\psi\big\|^{p-1}_{H^s(\mathbb{R}^n)}.\end{aligned}$$

Thus we the integral defining E converges at each time $t \in I$. We now show that E is indeed time independent. By approximating the data f in (1.1) by functions $f_k \in \mathcal{S}(\mathbb{R}^n)$ and with $\lim_k f_k = f$ in the $H^s(\mathbb{R}^n)$-norm, we reduce to the situation of

$$\psi \in C^\infty(I; H^\infty(\mathbb{R}^n))$$

We can henceforth differentiate with respect to t under the integral sign. To begin with, observe that

$$\partial_t\big((\frac{1}{2}\big|\nabla_x\psi\big|^2\big) = \mathrm{Re}\,\big(\nabla_x\psi_t \cdot \overline{\nabla_x\psi}\big)$$

and similarly

$$\begin{aligned}\partial_t\big(\frac{1}{p+1}|\psi|^{p+1}\big) &= \partial_t\big(\frac{1}{p+1}(\psi\bar{\psi})^{\frac{p+1}{2}}\big)\\ &= \frac{1}{2}\cdot\big(\psi_t\bar{\psi}+\psi\bar{\psi}_t\big)\cdot(\psi\bar{\psi})^{\frac{p+1}{2}-1}\\ &= \operatorname{Re}\big(\psi_t\bar{\psi}\big)\cdot\big|\psi\big|^{p-1}\end{aligned}$$

We conclude that

$$\begin{aligned}\frac{dE}{dt} &= \operatorname{Re}\int_{\mathbb{R}^n}\big(\nabla_x\psi_t\cdot\overline{\nabla_x\psi}\pm\big(\psi_t\bar{\psi}\big)\cdot\big|\psi\big|^{p-1}\big)\,dx\\ &= \operatorname{Re}\int_{\mathbb{R}^n}\big(-\psi_t\cdot\overline{\triangle\psi}\pm\big(\psi_t\bar{\psi}\big)\cdot\big|\psi\big|^{p-1}\big)\,dx\\ &= \operatorname{Re}\int_{\mathbb{R}^n}\psi_t\cdot\overline{\big(-\triangle\psi\pm\psi\cdot\big|\psi\big|^{p-1}\big)}\,dx\\ &= \operatorname{Re}\int_{\mathbb{R}^n}\psi_t\cdot\overline{i\psi_t}\,dx\\ &= 0.\end{aligned}$$

□

In a similar vein, we also have *mass conservation*:

Lemma 1.3. *For ψ as in the preceding lemma, we have that*

$$m := \int_{\mathbb{R}^n}\big|\psi(t,\cdot)\big|^2\,dx$$

is a conserved quantity.

Proof. As in the preceding proof we may reduce to smooth ψ. Then we compute

$$\begin{aligned}\frac{d}{dt}m &= 2\int_{\mathbb{R}^n}\operatorname{Re}\big(\psi_t\bar{\psi}\big)\,dx\\ &= 2\int_{\mathbb{R}^n}\operatorname{Im}\big(i\psi_t\bar{\psi}\big)\,dx\\ &= 2\int_{\mathbb{R}^n}\operatorname{Im}\big([\pm|\psi|^{p-1}\psi-\triangle\psi]\cdot\bar{\psi}\big)\,dx\\ &= 0.\end{aligned}$$

□

Using the preceding lemmas, we now show the following theorem in the case $n=1$ and the defocussing situation:

Theorem 1.4. *Assume $n=1$, $s\geq 1$, and $p\geq 1$. Then (1.1) with the $+$-sign on the right hand side admits a unique global solution*

$$\psi\in C^0(\mathbb{R};H^s(\mathbb{R}^n))$$

which depends continuously on the initial data f, in the sense that for any finite interval $I\subset\mathbb{R}$, the solution depends continuously on the data in the sense of

$$C^0(I;H^s(\mathbb{R}^n)).$$

Thus the problem is globally well-posed for these data.

Proof. By Theorem 2.1 there exists an open interval I centered at $t=0$ and a unique solution in $C^0(I;H^s(\mathbb{R}^n))$, whose length only depends on

$$\big\|f\big\|_{H^1(\mathbb{R})}.$$

Pick any $t_1 \in I$ with $\mathrm{dist}(t_1, I^c) < \frac{|I|}{4}$. By the preceding lemmas we know that (check the first inequality carefully)

$$\|\psi(t_1,\cdot)\|_{\dot{H}^1(\mathbb{R})} \leq C(1+\|f\|_{H^1(\mathbb{R})})^p,$$
$$\|\psi(t_1,\cdot)\|_{L^2_x} = \|f\|_{L^2_x},$$

whence

$$\|\psi(t_1,\cdot)\|_{H^1(\mathbb{R})} \leq D(1+\|f\|_{H^1(\mathbb{R})})^p$$

where D is a constant only depending on p.
Reiterating application of Theorem 2.1 allows us to extend the solution to the shifted interaval $I_1 := t_1 + I$, and so we now have a solution on

$$I \cup I_1.$$

This argument can be continued to construct a solution globally in time. We leave verification of continuous dependence on the data as an exercise. □

Remark 1.5. We observe that this argument does not say anything about the behavior of the solution when $t \to \pm\infty$, for example whether the solution scatters or not(meaning it approaches a solution of the free Schrodinger equation).

2. Scaling and regularity

The preceding argument gives no information for the case $n \geq 2$, since Theorem 2.1 from last lecture requires $s > \frac{n}{2} \geq 1$ Sobolev regularity of the data f, and we do not have a priori control over

$$\|\psi(t,\cdot)\|_{H^s(\mathbb{R}^n)}$$

for $s > 1$, at least as far as tan analogue of the simple lemma from the preceding section is concerned. It is then natural to ask whether we can *sharpen Theorem 2.1*, i. e. obtain local well-posedness for less regular data, and ideally of regularity $s \leq 1$, since then we could reiterate the preceding argument.
To understand this better, it pays to recall the sharp Sobolev embedding,

$$\|f\|_{L^p(\mathbb{R}^n)} \leq C \cdot \|f\|_{\dot{H}^s(\mathbb{R}^n)},\ s = \frac{p-2}{p}\cdot\frac{n}{2},\ \infty > p \geq 2.$$

To begin with, we note that the relation on the right between p and s is sharp, and cannot be replaced by another such relation. Indeed, this is easily seen by replacing f by its *re-scaled version*

$$f_\lambda(x) := f(\lambda x),\ \lambda > 0.$$

Then

$$\|f_\lambda\|_{L^p(\mathbb{R}^n)} = \lambda^{-\frac{n}{p}} \cdot \|f\|_{L^p(\mathbb{R}^n)},$$
$$\|f_\lambda\|_{\dot{H}^s(\mathbb{R}^n)} = \lambda^{s-\frac{n}{2}} \cdot \|f\|_{\dot{H}^s(\mathbb{R}^n)},$$

and so the inequality above can only hold for all $\lambda > 0$, $f \in \mathcal{S}(\mathbb{R}^n)$, provided

$$\lambda^{-\frac{n}{p}} = \lambda^{s-\frac{n}{2}},$$

which gives the previous relation between s, p, n.
We also note that s grows with p; this means *controlling lower Lebesgue norms $\|f\|_{L^p}$ requires less regularity.*

Now the strategy we used for proving Theorem 2. 1 consisted in controlling the Duhamel integral

$$\int_0^t S(t-s)\big(|\psi|^{p-1}\psi\big)(s,\cdot)\,ds$$

by means of

$$\Big\| \int_0^t S(t-s)\big(|\psi|^{p-1}\psi\big)(s,\cdot)\,ds\Big\|_{H^s} \leq |I|\cdot\big\||\psi|^{p-1}\psi\big\|_{L_t^\infty H^s(I\times\mathbb{R}^n)} \leq |I|\cdot\|\psi\|_{L^\infty(I\times\mathbb{R}^n)}^{p-1}\cdot\|\psi\|_{H^s(I\times\mathbb{R}^n)},$$

provided $t\in I$. We need a lot of regularity to control the term

$$\|\psi\|_{L^\infty(I\times\mathbb{R}^n)}^{p-1},$$

namely $s>\frac{n}{2}$. We know from the previous considerations that we could lower the regularity requirement if we replaced ∞ by some $q<\infty$ here, but it's not so clear how to deduce the required bound then.

A crucial insight, however, is that there is a slack in the previous inequality as regards the time integral. For example, we could try to bound

$$\big\||\psi|^{p-1}\psi\big\|_{L_t^1 H^s(I\times\mathbb{R}^n)}$$

instead of using

$$\big\||\psi|^{p-1}\psi\big\|_{L_t^\infty H^s(I\times\mathbb{R}^n)}.$$

This would mean that we would need a bound on

$$\|\psi\|_{L_t^{p-1}L_x^\infty},$$

and we may guess by using a similar scaling type argument that this ought to require less regularity of the initial data. The fact that a version of this reasoning can indeed be implemented has to do with the celebrated *Strichartz estimates* which we prove next.