BASIC LOCAL WELL-POSEDNESS OF NONLINEAR DISPERSIVE EQUATIONS IN $H^s(\mathbb{R}^n)$

1. Some basic nonlinear models

We shall now investigate the issue of constructing solutions for some simple nonlinear equations of both Schrodinger and wave type. Our basic tool kit shall be the Sobolev type spaces and linear propagators acting on them which we discussed in the last lecture. Let $p > 1$. Then the following equation is called the *nonlinear Schrodinger equation* or *NLS* for short:

$$i\psi_t + \triangle\psi = \pm|\psi|^{p-1}\psi,\ \psi(0,\cdot) = f. \tag{1.1}$$

The reason for using this nonlinearity on the right rather than $|\psi|^p$ or ψ^p when $p \in \mathbb{N}$ is related to the fact that (1.1) comes with a convenient conserved energy. We shall study this model on $\mathbb{R}^{1+n}$ where we let $n \geq 1$, and we shall for now also allow any $p > 1$. Later on we will see that there are important cases to distinguish depending on the relation between n and p. We note right away that even for such a simple model, at this point in time, the behavior of general solutions and for large values of n, p is mostly unknown.

We shall be interested in solutions of Sobolev regularity. To make sense of this also when the solutions are not necessarily smooth, we shall rely on

Definition 1.1. *Let $I \subset \mathbb{R}$ and open interval containing $t = 0$ and let $s \geq 0$, and $f \in H^s(\mathbb{R}^n)$. We say that $\psi \in C^0(I; H^s(\mathbb{R}^n)$ solves* (1.1) *in the Duhamel sense, provided*

$$|\psi|^{p-1}\psi \in L^1_{loc}(I; H^s(\mathbb{R}^n)),$$

and we have

$$\psi(t,\cdot) = S(t)f + (-i)\int_0^t S(t-s)\big(\pm|\psi|^{p-1}\psi(s,\cdot)\big)\,ds, \tag{1.2}$$

provided $t \in I$.

A completely analogous discussion applies to the pure power *nonlinear wave equation*, or *NLW* for short, which can be written as

$$\Box\psi = \pm|\psi|^{p-1}\psi,\ \Box = \partial_{tt} - \triangle,\ p > 1, \tag{1.3}$$

which we also study on $\mathbb{R}^{1+n}$. Here we shall seek to construct real-valued solutions with $\psi \in C^0(I; H^s(\mathbb{R}^s)) \cap C^1(I; H^{s-1}(\mathbb{R}^s))$, which satisfy the equation in the corresponding Duhamel sense. We do not spell out the exact definition, as it is analogous to the preceding one.
We note that for this model the behavior of solutions with large n, p is mostly unknown.

Our first order of the day shall be to establish the existence of local solutions in the first place, and to understand the somewhat subtle concept of *local well-posednesss* in Sobolev spaces.

2. Local well-posedness using the energy method

Our goal shall be to take advantage of Prop. 2. 2 from lecture 4 and the *Banach fixed point theorem* to infer the existence of a local solution of (1.1) and furthermore elucidate the concept of local well-posedness. For technical reasons, we shall limit the exponents p in order to avoid technical complications.

Theorem 2.1. *Let $f \in H^s(\mathbb{R}^n)$, $s > \frac{n}{2}$ and assume $p > 1$ is an odd integer. The exists an open interval $I \subset \mathbb{R}$ centered at $t = 0$ with $|I| = |I|(\|f\|_{H^s(\mathbb{R}^n)})$, such that* (1.1) *admits a unique solution in the Duhamel sense on $I \times \mathbb{R}^n$. This solution depends continuously on the initial data.*

The proof of this theorem rests on the following basic

Lemma 2.2. *Let f, g functions which are in $L^\infty(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$, $s \geq 0$. Then we have*

$$\big\|f \cdot g\big\|_{H^s(\mathbb{R}^n)} \leq C_{n,s} \cdot \Big(\big\|f\big\|_{L^\infty} \cdot \big\|g\big\|_{H^s} + \big\|g\big\|_{L^\infty} \cdot \big\|f\big\|_{H^s}\Big).$$

Proof. The case $s = 0$ is an immediate consequence of Holder's inequality, so we assume now that $s > 0$. We shall use Fourier localization techniques. To begin with, given $l \in \mathbb{Z}$, introduce the frequency localization operators

$$P_{<l} = \sum_{k<l} P_k,\ P_{[l-10,l+9]} = \sum_{k=l-10}^{l+9} P_k,\ P_{\geq l} = 1 - P_{<l}.$$

Setting $l < 0$ I leave to you to check that

$$\big\|P_{<0}(f \cdot g)\big\|_{H^s} \leq C \cdot \big\|f\big\|_{L^\infty} \cdot \big\|g\big\|_{H^s}.$$

Now fix $l \in \mathbb{N}$, and *decompose* the frequency localized product as follows:

$$P_l\big(f \cdot g\big) = P_l\big(P_{<l-10} f \cdot g\big) + P_l\big(P_{[l-10,l+9]} f \cdot g\big) + P_l\big(P_{\geq l+10} f \cdot g\big). \tag{2.1}$$

Then the following simple relations can be proved by computing the Fourier transform of the products (resulting in convolution integrals):

$$\begin{aligned} P_l\big(P_{<l-10} f \cdot g\big) &= P_l\big(P_{<l-10} f \cdot P_{[l-5,l+5]} g\big), \\ P_l\big(P_{\geq l+10} f \cdot g\big) &= \sum_{\substack{k_1 \geq l+10 \\ |k_1-k_2| \leq 5}} P_l\big(P_{k_1} f \cdot P_{k_2} g\big) \end{aligned} \tag{2.2}$$

Now we can easily bound the first two terms in (2.1):

$$\begin{aligned} \big\|P_l\big(P_{<l-10} f \cdot g\big)\big\|_{H^s} &\leq C \cdot 2^{sl} \cdot \big\|P_{<l-10} f \cdot g\big\|_{L^2} \\ &\leq C \cdot 2^{sl} \cdot \big\|P_{<l-10} f\big\|_{L^\infty} \cdot \big\|P_{[l-5,l+5]} g\big\|_{L^2} \\ &\leq \widetilde{C} \cdot \big\|f\big\|_{L^\infty} \cdot \big\|P_{[l-5,l+5]} g\big\|_{H^s}, \end{aligned}$$

where we have used the inequalities

$$\big\|P_{<l-10} f\big\|_{L^\infty} \leq C\big\|f\big\|_{L^\infty},\ 2^{sl} \cdot \big\|P_{[l-5,l+5]} g\big\|_{L^2} \leq C \cdot \big\|P_{[l-5,l+5]} g\big\|_{H^s},$$

which are left as exercise.
Similarly, we have

$$\begin{aligned} \big\|P_l\big(P_{[l-10,l+9]} f \cdot g\big)\big\|_{H^s} &\leq C \cdot 2^{sl} \cdot \big\|P_{[l-10,l+9]} f\big\|_{L^2} \cdot \big\|g\big\|_{L^\infty} \\ &\leq \widetilde{C} \cdot \big\|P_{[l-10,l+9]} f\big\|_{H^s} \cdot \big\|g\big\|_{L^\infty}. \end{aligned}$$

It remains to bound the second term in (2.2):

$$\begin{aligned} \Big\|\sum_{\substack{k_1 \geq l+10 \\ |k_1-k_2| \leq 5}} P_l\big(P_{k_1} f \cdot P_{k_2} g\big)\Big\|_{H^s} &\leq C \cdot 2^{sl} \cdot \sum_{\substack{k_1 \geq l+10 \\ |k_1-k_2| \leq 5}} \big\|P_{k_1} f\big\|_{L^\infty} \cdot \big\|P_{k_2} g\big\|_{L^2} \\ &\leq \widetilde{C} \cdot \big\|f\big\|_{L^\infty} \cdot \sum_{\substack{k_1 \geq l+10 \\ |k_1-k_2| \leq 5}} 2^{(l-k_2)s} \cdot \big\|P_{k_2} g\big\|_{H^s}. \end{aligned}$$

We can now complete the proof of the lemma. Note that

$$\begin{aligned}\|f\cdot g\|_{H^s} &\leq C\cdot\Big(\sum_l \|P_l(f\cdot g)\|_{H^s}^2\Big)^{\frac{1}{2}}\\ &\leq C_1\cdot\Big(\sum_l \|P_l(P_{<l-10}f\cdot g)\|_{H^s}^2\Big)^{\frac{1}{2}}+\Big(\sum_l \|P_l(P_{[l-10,l+9]}f\cdot g)\|_{H^s}^2\Big)^{\frac{1}{2}}\\ &\quad+\Big(\sum_l \|P_l(P_{[\geq l+10]}f\cdot g)\|_{H^s}^2\Big)^{\frac{1}{2}}\end{aligned}$$

Then the considerations above allow us to bound

$$\begin{aligned}\Big(\sum_l \|P_l(P_{<l-10}f\cdot g)\|_{H^s}^2\Big)^{\frac{1}{2}} &\leq \widetilde{C}\cdot\|f\|_{L^\infty}\cdot\Big(\sum_l \|P_{[l-5,l+5]}g)\|_{H^s}^2\Big)^{\frac{1}{2}}\\ &\leq C_2\cdot\|f\|_{L^\infty}\cdot\|g\|_{H^s}.\end{aligned}$$

The bound

$$\Big(\sum_l \|P_l(P_{[l-10,l+9]}f\cdot g)\|_{H^s}^2\Big)^{\frac{1}{2}}\leq C_3\cdot\|f\|_{H^s}\cdot\|g\|_{L^\infty}$$

is analogous.
Finally, we need to bound the double sum

$$\Big(\sum_l\big(\sum_{\substack{k_1\geq l+10\\|k_1-k_2|\leq 5}}2^{(l-k_2)s}\cdot\|P_{k_2}g\|_{H^s}\big)^2\Big)^{\frac{1}{2}},$$

which requires a trick, namely application of the Cauchy-Schwarz inequality:

$$\begin{aligned}\big(\sum_{\substack{k_1\geq l+10\\|k_1-k_2|\leq 5}}2^{(l-k_2)s}\cdot\|P_{k_2}g\|_{H^s}\big)^2 &\leq C_4\big(\sum_{k_2\geq l+5}2^{(l-k_2)s}\cdot\|P_{k_2}g\|_{H^s}^2\big)\cdot\big(\sum_{k_2\geq l+5}2^{(l-k_2)s}\big)\\ &\leq C_5\cdot\sum_{k_2\geq l+5}2^{(l-k_2)s}\cdot\|P_{k_2}g\|_{H^s}^2.\end{aligned}$$

But then applying the summation over $l\in\mathbb{N}$ we infer

$$\begin{aligned}&\Big(\sum_l\big(\sum_{\substack{k_1\geq l+10\\|k_1-k_2|\leq 5}}2^{(l-k_2)s}\cdot\|P_{k_2}g\|_{H^s}\big)^2\Big)^{\frac{1}{2}}\\ &\leq\Big(C_5\cdot\sum_l\sum_{k_2\geq l+5}2^{(l-k_2)s}\cdot\|P_{k_2}g\|_{H^s}^2\Big)^{\frac{1}{2}}\\ &\leq C_6\cdot\|g\|_{H^s}.\end{aligned}$$

Using this and the preceding considerations we infer

$$\Big(\sum_l \|P_l(P_{[\geq l+10]}f\cdot g)\|_{H^s}^2\Big)^{\frac{1}{2}}\leq C_7\cdot\|f\|_{L^\infty}\cdot\|g\|_{H^s}.$$

□

Proof. (Theorem 2.1) This will be based on the Banach fixed point principle. Given $f\in H^s(\mathbb{R}^n)$, consider the map from

$$C^0(I,\,H^s(\mathbb{R}^n))$$

to itself given by

$$\psi \longrightarrow T(\psi) := \int_0^t S(t-s)\big(\pm|\psi|^{p-1}\psi\big)\,ds + S(t)(f)$$

Then we claim

Lemma 2.3. *The map T maps $C^0(I,\,H^s(\mathbb{R}^n))$ into itself. It is a contraction on*

$$B_{2\|f\|_{H^s}}(0) \subset C^0(I,\,H^s(\mathbb{R}^n))$$

where $B_a(p)$ denotes the ball of radius a centered at a (in $C^0(I,\,H^s(\mathbb{R}^n))$), provided

$$|I|$$

is sufficiently small depending on $\|f\|_{H^s(\mathbb{R}^n)}$.

Proof. (lemma) Writing

$$|\psi|^{p-1}\psi = (\psi\bar{\psi})^{\frac{p-1}{2}} \cdot \psi$$

where $\frac{p-1}{2} \in \mathbb{N}$ by assumption, we infer by inductively applying Lemma 2.2 as well as Sobolev's embedding that (at fixed time t)

$$\Big\||\psi|^{p-1}\psi\Big\|_{H^s(\mathbb{R}^n)} \leq C_{p,n}\|\psi\|^p_{H^s(\mathbb{R}^n)}.$$

Taking advantage of the fact that $S(t)$ is an isometry on $H^s(\mathbb{R}^n)$, we find that

$$\begin{aligned}
&\Big\|\int_0^t S(t-s)\big(\pm|\psi|^{p-1}\psi\big)(s,\cdot)\,ds + S(t)(f)\Big\|_{H^s(\mathbb{R}^n)} \\
&\leq \Big\|S(t-s)\big(\pm|\psi|^{p-1}\psi\big)(s,\cdot)\Big\|_{L^1_s H^s(I\times\mathbb{R}^n)} + \|f\|_{H^s(\mathbb{R}^n)} \\
&\leq C\cdot|I|\cdot\|\psi\|^p_{C^0(I,\,H^s(\mathbb{R}^n))} + \|f\|_{H^s(\mathbb{R}^n)}.
\end{aligned}$$

It easily follows that T maps $C^0(I,\,H^s(\mathbb{R}^n))$ into itself. Furthermore, if $\psi_1,\psi_2 \in C^0(I,\,H^s(\mathbb{R}^n))$, then

$$\begin{aligned}
&\Big\|T(\psi_1)-T(\psi_2)\Big\|_{L^\infty_t H^s(I\times\mathbb{R}^n)} \\
&\leq D_{p,s,n}\cdot|I|\cdot\|\psi_1-\psi_2\|_{L^\infty_t H^s(I\times\mathbb{R}^n)}\cdot\big(\|\psi_1\|_{L^\infty_t H^s(I\times\mathbb{R}^n)}+\|\psi_2\|_{L^\infty_t H^s(I\times\mathbb{R}^n)}\big)^{p-1}
\end{aligned}$$

It follows from the preceding two estimates that if we impose the condition

$$|I| \leq \frac{\|f\|_{H^s(\mathbb{R}^n)}}{C\cdot(2\|f\|_{H^s(\mathbb{R}^n)})^p} = (2^pC)^{-1}\cdot\|f\|^{-(p-1)}_{H^s(\mathbb{R}^n)}$$

the map T sends $B_{2\|f\|_{H^s}}(0)$ into itself, and further if

$$|I| \leq \frac{1}{2}\cdot D^{-1}_{p,s,n}\cdot(4\|f\|_{H^s(\mathbb{R}^n)})^{-(p-1)},$$

we have that

$$\Big\|T(\psi_1)-T(\psi_2)\Big\|_{L^\infty_t H^s(I\times\mathbb{R}^n)} \leq \frac{1}{2}\cdot\|\psi_1-\psi_2\|_{L^\infty_t H^s(I\times\mathbb{R}^n)}.$$

□

The preceding lemma and Banach's fixed point principle implies that there is a unique fixed point $\psi \in \overline{B_{2\|f\|_{H^s}}(0)} \subset C^0(I,\,H^s(\mathbb{R}^n))$ for T provided

$$|I| \leq \min\{\frac{1}{2}\cdot D^{-1}_{p,s,n}\cdot(4\|f\|_{H^s(\mathbb{R}^n)})^{-(p-1)},\,(2^pC)^{-1}\cdot\|f\|^{-(p-1)}_{H^s(\mathbb{R}^n)}\}$$

A simple continuity argument then implies that ψ is indeed the unique solution for (1.2) in $C^0(I,\,H^s(\mathbb{R}^n))$ with I satisfying the preceding condition. The continuous dependence of ψ on f is left as an exercise. □

An important additional feature of the solution just constructed in the preceding theorem is the fact that *additional regularity of the initial data is preserved.* In particular, if $f \in C^\infty(\mathbb{R}^n)$, then so is ψ.

Proposition 2.4. *Let (I, s, ψ) be as in the preceding theorem, and assume that the data $f \in H^{s_1}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ where $s_1 > s$. Then we have*

$$\psi \in C^0(I,\, H^{s_1}(\mathbb{R}^n)).$$

Remark 2.5. The point here is that the length of I only depends on $\|f\|_{H^s}$, but the solution preserves the additional regularity of f.

Proof. The fixed point ψ for the map T in Lemma 2.3 is obtained from the Banach fixed point theorem, and recalling the proof of the latter, ψ is the limit of a sequence of functions ψ_i, $i = 0, 1, 2, \ldots$, where

$$\psi_{i+1}(t, \cdot) = \int_0^t S(t-s)\big(\pm|\psi_i|^{p-1}\psi_i\big)\, ds + S(t)(f),\ i = 0, 1, 2, \ldots, \tag{2.3}$$

while the 'zeroth' iterate ψ_0 is simply the linear propagator

$$\psi_0(t, \cdot) = S(t)(f).$$

Now if $f \in H^{s_1}(\mathbb{R}^n)$, $s_1 > s$, then

$$\psi_0 \in C^0\big(\mathbb{R}; H^{s_1}(\mathbb{R}^n)\big),$$

by using the results of Lecture 4. We now show inductively that replacing I by an interval $\widetilde{I}$ centered at $t = 0$ and whose length only depends on

$$\|f\|_{H^s(\mathbb{R}^n)},\ n,\ ,s,\ s_1,$$

the sequence $\{\psi_i\}_{i\geq 1}|$ converges in $C^0\big(\widetilde{I}; H^{s_1}(\mathbb{R}^n)\big)$. The key here is that the length of $\widetilde{I}$ only depends on $\|f\|_{H^s}$, and not on $\|f\|_{H^{s_1}}$. To see this, we use Lemma 2.2. This furnishes the bound

$$\Big\|\big(\pm|\psi_i|^{p-1}\psi_i\big)\Big\|_{H^{s_1}(\mathbb{R}^n)} \leq C_{s_1,n,p} \cdot \|\psi_i\|_{L^\infty}^{p-1} \cdot \|\psi_i\|_{H^{s_1}(\mathbb{R}^n)},$$

as well as the difference bound

$$\Big\|\big(\pm|\psi_i|^{p-1}\psi_i\big) - \big(\pm|\psi_{i-1}|^{p-1}\psi_{i-1}\big)\Big\|_{H^{s_1}(\mathbb{R}^n)} \leq D_{s_1,n,p} \cdot \big(\|\psi_i\|_{L^\infty(\mathbb{R}^n)}^{p-1} + \|\psi_{i-1}\|_{L^\infty(\mathbb{R}^n)}^{p-1}\big) \cdot \|\psi_i - \psi_{i-1}\|_{H^{s_1}(\mathbb{R}^n)}$$

Using the Sobolev embedding, we infer that

$$\|\psi_i\|_{L^\infty(\mathbb{R}^n)} \leq E_{n,s} \cdot \|\psi_i\|_{H^s(\mathbb{R}^n)}.$$

Now we know that $\|\psi_i(t, \cdot)\|_{H^s(\mathbb{R}^n)} \leq F(s, n, I, \|f\|_{H^s(\mathbb{R}^n)})$, as long as $t \in I$. If we now choose

$$|\widetilde{I}| \leq \frac{1}{[2C_{s_1,n,p} + 4D_{s_1,n,p}] \cdot F^{p-1}(s, n, I, \|f\|_{H^s(\mathbb{R}^n)})},$$

we infer inductively that

$$\|\psi_i\|_{L_t^\infty H^{s_1}(\widetilde{I}\times\mathbb{R}^n)} \leq 2\|f\|_{H^{s_1}(\mathbb{R}^n)},$$

$$\|\psi_{i+1} - \psi_i\|\|_{L_t^\infty H^{s_1}(\widetilde{I}\times\mathbb{R}^n)} \leq \frac{1}{2} \cdot \|\psi_i - \psi_{i-1}\|\|_{L_t^\infty H^{s_1}(\widetilde{I}\times\mathbb{R}^n)}.$$

It follows that $\psi_i \longrightarrow \psi$ in $C^0(\widetilde{I},\, H^{s_1}(\mathbb{R}^n))$. To get the desired conclusion on all of I instead of just on the shorter subinterval $\widetilde{I}$, we repeat the preceding argument a finite number of times. In fact, we can cover I with intervals

$$\widetilde{I} = \widetilde{I}_1, \widetilde{I}_2, \ldots, \widetilde{I}_N,$$

where $|\widetilde{I}_j| = |\widetilde{I}|$, and

$$N = \lfloor\frac{|I|}{|\widetilde{I}|}\rfloor + 1.$$

□

Generally speaking, we have established that (1.1) is *strongly locally well-posed* in $H^s(\mathbb{R}^n)$, $s > \frac{n}{2}$, in the sense of *existence of local solutions in* $C^0(I, H^s(\mathbb{R}^n))$*, uniqueness, (uniformly) continuous dependence on the data (actually, even smooth dependence), and preservation of higher regularity.*