SOLUTIONS OF DISPERSIVE EQUATIONS IN $H^s(\mathbb{R}^n)$

1. Weak solutions of linear wave and Schrodinger equations

Using the explicit formulas (using the frequency representation) for the solutions of the linear homogeneous equations

$$\Box\psi = 0,\ \psi[0] = (f, g),\ \Box = \partial_{tt} - \triangle_{R^n},$$
$$i\psi_t + \triangle\psi = 0,\ \psi(0) = h,$$

it is natural to consider data which live in the Sobolev spaces H^s. More specifically, in the case of the wave equation we shall take data

$$(f, g) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$$

where we shall restrict $s \geq 1$, while for the Schrodinger equation we shall assume

$$h \in H^s(\mathbb{R}^n),\ s \geq 0.$$

In the sequel we shall frequently use the following

Lemma 1.1. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space (over $\mathbb{C}$ or $\mathbb{R}$) and let*

$$F : I \to H$$

a continuous function, where $I \subset \mathbb{R}$ is an open interval. Then fixing $t_0 \in I$, there exists a unique function

$$\int_{t_0}^{t} F(s)\, ds : I \to H$$

characterized by the property that for each $v \in H$, we have

$$\langle \int_{t_0}^{t} F(s)\, ds, v\rangle = \int_{t_0}^{t} \langle F(s), v\rangle\, ds.$$

The function $\int_{t_0}^{t} F(s)\, ds$ again depends continuously on $t \in I$.

Proof. For $t \in I$ introduce $T_t \in H^*$ by means of

$$T_t(v) := \int_{t_0}^{t} \langle F(s), v\rangle\, ds.$$

This is indeed an element in H^* since

$$\big\|T_t(v)\big\| \leq \|v\| \cdot \int_{t_0}^{t} \|F(s)\|\, ds =: C_t \cdot \|v\|.$$

By the Riesz representation theorem, there is a unique $w_t \in H$ with the property that

$$T_t(v) = \langle w_t, v\rangle.$$

Then we set

$$\int_{t_0}^{t} F(s)\, ds := w_t.$$

The fact that this function depends continuously on $t \in I$ is left as an exercise. $\square$

We then make the

Definition 1.2. *Given a Hilbert space H (over $\mathbb{C}$ or $\mathbb{R}$), we call a function*

$$F : I \to H,\ I \subset \mathbb{R} \text{ open interval}$$

continuously differentiable, provided there exists a continuous function

$$G : I \to H$$

with the property that

$$F(t) = \int_{t_0}^{t} G(s)\, ds + v$$

for some $t_0 \in I$ and some $v \in H$. We then write

$$F \in C^1(I; H),\ F_t = G.$$

We can now analyse the solutions of the wave and Schrodinger equations:

Proposition 1.3. *Assume that $h \in H^s(\mathbb{R}^n)$, $s \geq 0$. Then the function*

$$t \to (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(-t|\xi|^2 + x\cdot\xi)} \widehat{h}(\xi)\, d\xi$$

defines a continuous function from $\mathbb{R}$ into $H^s(\mathbb{R}^n)$, which is in fact an isometry there for each t. If $(f, g) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$, $s \geq 1$, the function

$$\begin{aligned} t \to F(t) :=& (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(t|\xi| + x\cdot\xi)} \cdot \frac{1}{2}[\widehat{f}(\xi) + \frac{1}{i|\xi|}\widehat{g}(\xi)]\, d\xi \\ &+ (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(-t|\xi| + x\cdot\xi)} \cdot \frac{1}{2}[\widehat{f}(\xi) - \frac{1}{i|\xi|}\widehat{g}(\xi)]\, d\xi \end{aligned}$$

is in

$$C^0(\mathbb{R}; H^s(\mathbb{R}^n)) \cap C^1(\mathbb{R}; H^{s-1}(\mathbb{R}^n)).$$

Proof. For the Schrodinger part, observe that for fixed $t \in R$

$$\begin{aligned} \Big\|(2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(-t|\xi|^2 + x\cdot\xi)} \widehat{h}(\xi)\, d\xi\Big\|_{H^s(\mathbb{R}^n)} &= \big\|(1 + |\xi|^2)^{\frac{s}{2}} \cdot e^{-it|\xi|^2}\widehat{h}(\xi)\big\|_{L^2(\mathbb{R}^n)} \\ &= \big\|(1 + |\xi|^2)^{\frac{s}{2}} \cdot \widehat{h}(\xi)\big\|_{L^2(\mathbb{R}^n)} \\ &= \big\|h\big\|_{H^s(\mathbb{R}^n)}. \end{aligned}$$

Moreover, for $t, t' \in \mathbb{R}$, we have

$$\begin{aligned} &\lim_{t' \to t}\Big\|(2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(-t'|\xi|^2 + x\cdot\xi)} \widehat{h}(\xi)\, d\xi - (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(-t|\xi|^2 + x\cdot\xi)} \widehat{h}(\xi)\, d\xi\Big\|_{H^s(\mathbb{R}^n)} \\ &= \lim_{t' \to t} \big\||(1 + |\xi|^2)^{\frac{s}{2}} \cdot (e^{-it|\xi|^2} - e^{-it'|\xi|^2})\widehat{h}(\xi)\big\|_{L^2(\mathbb{R}^n)} \\ &= 0 \end{aligned}$$

where for the last equality we have used the dominated convergence theorem.

For the wave propagator one has to carefully observe that the function

$$t \to (2\pi)^{-n} \int_{\mathbb{R}^n} \left(e^{i(t|\xi| + x\cdot\xi)} - e^{i(-t|\xi| + x\cdot\xi)}\right) \cdot \frac{1}{i|\xi|}\widehat{g}(\xi)]\, d\xi$$

maps continuously into $H^s(\mathbb{R}^n)$. where one exploits that the function

$$e^{i(t|\xi| + x\cdot\xi)} - e^{i(-t|\xi| + x\cdot\xi)}$$

vanishes at $\xi = 0$. Moreover, it is easy to check that

$$\begin{aligned} F_t =&(2\pi)^{-n}\int_{\mathbb{R}^n}(i|\xi|)e^{i(t|\xi|+x\cdot\xi)}\cdot\frac{1}{2}[\widehat{f}(\xi)+\frac{1}{i|\xi|}\widehat{g}(\xi)]\,d\xi\\ &+(2\pi)^{-n}\int_{\mathbb{R}^n}(-i|\xi|)e^{i(-t|\xi|+x\cdot\xi)}\cdot\frac{1}{2}[\widehat{f}(\xi)-\frac{1}{i|\xi|}\widehat{g}(\xi)]\,d\xi \end{aligned}$$

where the function on the right maps continuously into $H^{s-1}(\mathbb{R}^n)$.

□

The preceding proposition allows us to define the solution of the linear Schrodinger and wave equations by means of the explicit propagator of the data, but this still leaves the question in what sense these expressions are actually 'solutions' of the original equations. Hence we make the following definition, which gives the *weak intrepretation* of solutions of our PDE:

Definition 1.4. *We say that a function $\psi \in C^0(\mathbb{R}_{\geq 0}; H^s(\mathbb{R}^n))$, $s \geq 0$, is a weak solution of*

$$i\psi_t + \triangle\psi = 0,\ \psi(0) = f \in H^s(\mathbb{R}^n),$$

on $\mathbb{R}_{\geq 0}\times\mathbb{R}^n$, provided for each test function $\zeta(\cdot,\cdot) \in C_0^\infty(\mathbb{R}_{\geq 0}\times\mathbb{R}^n)$, we have

$$\int_0^\infty\int_{\mathbb{R}^n}\big(-i\psi\zeta_t + \psi\triangle\zeta\big)\,dxdt = i\int_{\mathbb{R}^n} f(x)\zeta(0,x)\,dx.$$

Similarly, we say that $\psi \in C^0(\mathbb{R}_{\geq 0}; H^s(\mathbb{R}^n)) \cap C^1(\mathbb{R}_{\geq 0}; H^{s-1})(\mathbb{R}^n))$, $s \geq 1$, is a weak solution of

$$\Box\psi = 0,\ \psi[0] = (f,g) \in H^s(\mathbb{R}^n)\times H^{s-1}(\mathbb{R}^n)$$

on $\mathbb{R}_{\geq 0}\times\mathbb{R}^n$, provided that for each test function $\zeta(\cdot,\cdot) \in C_0^\infty(\mathbb{R}_{\geq 0}\times\mathbb{R}^n)$, we have

$$\int_0^\infty\int_{\mathbb{R}^n}\psi\cdot\Box\zeta\,dxdt = \int_{\mathbb{R}^n} g(x)\zeta(0,x)\,dx - \int_{\mathbb{R}^n} f(x)\zeta_t(0,x)\,dx,\ \forall T \in \mathbb{R}$$

This definition is meaningful for us due to the following

Proposition 1.5. *The expressions given in Prop. 1.3 are weak solutions of the Schrodinger, resp. the wave equation.*

Proof. Given $h \in H^s(\mathbb{R}^n)$, $s \geq 0$, an initial datum for the Schrodinger equation, pick a sequence $\{h_k\}_{k\geq 1} \subset \mathcal{S}(\mathbb{R}^n)$ converging to h in the $H^s(\mathbb{R}^n)$-norm. Then the functions

$$\psi_k(t,x) := (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(-t|\xi|^2+x\cdot\xi)}\cdot\widehat{h}_k(\xi)\,d\xi$$

solve the linear Schrodinger equation and moreover the identity

$$\int_0^\infty\int_{\mathbb{R}^n}\big(-i\psi_k\zeta_t + \psi_k\triangle\zeta\big)\,dxdt = i\int_{\mathbb{R}^n} h_k(x)\zeta(0,x)\,dx$$

follows by simple integration by parts. But since

$$\begin{aligned} &\lim_{k\to\infty}\int_0^\infty\int_{\mathbb{R}^n}\big(-i\psi_k\zeta_t + \psi_k\triangle\zeta\big)\,dxdt = \int_0^\infty\int_{\mathbb{R}^n}\big(-i\psi\zeta_t + \psi\triangle\zeta\big)\,dxdt,\\ &\lim_{k\to\infty} i\int_{\mathbb{R}^n} h_k(x)\zeta(0,x)\,dx = i\int_{\mathbb{R}^n} h(x)\zeta(0,x)\,dx, \end{aligned}$$

the assertion of the proposition for the Schrodinger equation follows.
The proof for the linear wave equation is similar.

□

2. Towards nonlinear problems; solution of linear inhomogeneous problems in Sobolev spaces

So far we have considered *homogeneous linear dispersive PDE*, i. e. with vanishing right hand side, but now is the time to also consider *inhomogeneous linear equations*, on our way towards nonlinear problems. To begin with, consider the linear inhomogeneous Schrodinger equation

$$i\psi_t + \triangle\psi = F,\, \psi(0) = f \tag{2.1}$$

on $\mathbb{R}^{1+n}$. Here we shall assume that

$$F \in C^0(\mathbb{R}; H^s(\mathbb{R}^n)),\, f \in H^s(\mathbb{R}^n),$$

and in fact it is enough to assume that

$$F \in L^1_{loc}(\mathbb{R}; H^s).$$

We can then solve this problem in the following sense:

Definition 2.1. *We say that a function $\psi \in C^0(\mathbb{R}_{\geq 0}; H^s(\mathbb{R}^n))$, $s \geq 0$, is a weak solution of*

$$i\psi_t + \triangle\psi = F,\, \psi(0) = f \in H^s(\mathbb{R}^n),\, F \in C^0(\mathbb{R}; H^s(\mathbb{R}^n)),$$

on $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$, provided for each test function $\zeta(\cdot, \cdot) \in C_0^\infty(\mathbb{R}_{\geq 0} \times \mathbb{R}^n)$, we have

$$\int_0^\infty \int_{\mathbb{R}^n} \big(- i\psi\zeta_t + \psi\triangle\zeta\big)\, dxdt = \int_0^\infty \int_{\mathbb{R}^n} F\zeta\, dxdt + i\int_{\mathbb{R}^n} f(x)\zeta(0,x)\, dx.$$

We then have the following

Proposition 2.2. *The problem* (2.1) *admits the weak solution*

$$\psi(t,\cdot) = S(t)f + (-i)\int_0^t S(t-s)F(s)\, ds,$$

where $S(t)$ denotes the homogeneous Schrodinger propagator

$$\big(S(t)f\big)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|^2)}\widehat{f}(\xi)\, d\xi.$$

The preceding formula for ψ is also referred to as Duhamel formula. The solution satisfies the following bound:

$$\big\|\psi\big\|_{L_t^\infty H^s(\mathbb{R}^n)} \leq \big\|f\big\|_{H^s(\mathbb{R}^n)} + \big\|F\big\|_{L_t^1 H^s(\mathbb{R}^n)}.$$

Proof. Note that the function

$$u \to S(t-u)F(u)$$

is a continuous H^s valued function, and so the integral can be defined in the sense of Lemma 1.1. Moreover, choosing a sequence

$$\{F_k\}_{k\geq} \subset C^0\big(\mathbb{R}_{\geq 0}; \mathcal{S}(\mathbb{R}^n)\big)$$

such that $F_k \longrightarrow F$ in the topology of $C^0_{loc}(\mathbb{R}; H^s(\mathbb{R}^n))$, and further $\{f_k\}_{k\geq 1} \subset \mathcal{S}(\mathbb{R}^n)$ with

$$f_k \to f$$

in the sense of $H^s(\mathbb{R}^n)$, we obtain

$$\psi = \lim_{k\to\infty} \psi_k$$

where

$$\psi_k(t,\cdot) = S(t)f_k + (-i)\int_0^t S(t-s)F_k(s)\, ds$$

and the limit is in the $C^0_{loc}(\mathbb{R}; H^s(\mathbb{R}^n))$-topology. Then the fact that

$$\int_0^\infty \int_{\mathbb{R}^n} \big(- i\psi_k\zeta_t + \psi_k\triangle\zeta\big)\, dxdt = \int_0^\infty \int_{\mathbb{R}^n} F_k\zeta\, dxdt + i\int_{\mathbb{R}^n} f_k(x)\zeta(0,x)\, dx$$

for $\zeta(\cdot, \cdot) \in C_0^\infty(\mathbb{R}_{\geq 0} \times \mathbb{R}^n)$ follows from the fact that

$$i\psi_{k,t} + \triangle\psi_k = F_k,\, \psi(0) = f_k$$

in the classical pointwise sense as well as integration against ζ and integration by parts. The relation

$$\int_0^\infty \int_{\mathbb{R}^n} \big(- i\psi\zeta_t + \psi\triangle\zeta\big)\, dxdt = \int_0^\infty \int_{\mathbb{R}^n} F\zeta\, dxdt + i\int_{\mathbb{R}^n} f(x)\zeta(0,x)\, dx$$

follows by passing to the limit.
The final inequality of the proposition is a consequence of the fact that $S(t)$ is an isometry in $H^s(\mathbb{R}^n)$, as well as the triangle inequality. □

We can make analogous observations for the inhomogeneous wave

Definition 2.3. *We say that a function*

$$\psi \in C^0(\mathbb{R}_{\geq 0}; H^s(\mathbb{R}^n)) \cap C^1(\mathbb{R}_{\geq 0}; H^{s-1}(\mathbb{R}^n)), s \geq 1$$

is a weak solution of

$$\Box\psi = F,\ \psi(0) = (f, g) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n),\ F \in C^0(\mathbb{R}; H^{s-1}(\mathbb{R}^n)), \tag{2.2}$$

on $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$*, provided for each test function* $\zeta(\cdot,\cdot) \in C_0^\infty(\mathbb{R}_{\geq 0} \times \mathbb{R}^n)$*, we have*

$$\int_0^\infty \int_{\mathbb{R}^n} \psi\Box\zeta\, dxdt = \int_0^\infty \int_{\mathbb{R}^n} F\zeta\, dxdt + \int_{\mathbb{R}^n} g(x)\zeta(0,x)\, dx - \int_{\mathbb{R}^n} f(x)\zeta_t(0,x)\, dx.$$

Then we have the following analogue of the preceding Prop. 2.2:

Proposition 2.4. *The problem* (2.2) *admits the weak solution*

$$\psi(t,\cdot) = S(t)(f,g) + \int_0^t U(t-s)F(s)\, ds,$$

where $S(t)$ *denotes the homogeneous wave propagator given in Proposition 1.3, while*

$$\big(U(t)F\big)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_0^t e^{ix\cdot\xi} \cdot \frac{\sin[(t-s)|\xi|]}{|\xi|} \widehat{F}(s,\xi)\, dsd\xi.$$

The preceding formula for ψ *is also referred to as Duhamel formula.*

The proof is left as an exercise.