# FUNCTIONS OF SOBOLEV REGULARITY; THE SPACES $H^s(\mathbb{R}^n)$

## 1. Definition and basic properties of the spaces $H^s(\mathbb{R}^n)$

So far we have restricted solutions to our linear dispersive models to highly regular function spaces, such as $\mathcal{S}(\mathbb{R}^n)$. It is however both very natural and important to consider a much larger class of solutions, namely those in a particular class of Sobolev spaces, $H^s$. Here we define these function spaces and give some basic properties:

**Definition 1.1.** *We say a function $f \in L^2(\mathbb{R}^n)$ belongs to the Sobolev space $H^s(\mathbb{R}^n)$, $s \geq 0$, provided the function*

$$\widehat{f}(\xi) \cdot (1 + |\xi|^2)^{\frac{s}{2}} \in L^2(\mathbb{R}^n).$$

*If so, we introduce the norm*

$$\big\|f\big\|_{H^s(\mathbb{R}^n)} := \big\|\widehat{f}(\xi) \cdot (1 + |\xi|^2)^{\frac{s}{2}}\big\|_{L^2(\mathbb{R}^n)}.$$

*We also define the homogeneous Sobolev norm*

$$\big\|f\big\|_{\dot{H}^s(\mathbb{R}^n)} := \big\|\widehat{f}(\xi) \cdot |\xi|^s\big\|_{L^2(\mathbb{R}^n)}.$$

*Remark* 1.2. We stress that $s \geq 0$ can take any non-negative real number (one can also introduce the Sobolev spaces for negative indices, but we will mostly avoid these in this course). In the special case that $s = k \in \mathbb{N}_{\geq 0}$, we have an alternative definition of the space $H^k$, namely

$$H^k = \{f \in L^2(\mathbb{R}^n) \,|\, \Pi_{j=1}^n \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}} f \in L^2(\mathbb{R}^n)\, \forall \underline{\alpha} \in \mathbb{N}^n \text{ with } |\underline{\alpha}| \leq k\}$$

In the preceding definition we use the notation

$$\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}$$

as well as

$$|\underline{\alpha}| := \sum_{j=1}^n |\alpha_j|,$$

so this norm *differs* from the usual Euclidean one for multi indices.

We mention the following basic proposition without proof:

**Proposition 1.3.** *The sets $H^s$ form vector spaces (over $\mathbb{C}$). Furthermore, letting*

$$\langle f,\, g\rangle_{H^s} := \int_{\mathbb{R}^n} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} \cdot (1 + |\xi|^2)^s \, d\xi,$$

*whence $\big\|f\big\|_{H^s} = \sqrt{\langle f, f\rangle_{H^s}}$, the space $H^s$ becomes a Hilbert space. The space $\mathcal{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ is a dense subspace.*

We shall soon see that if $s$ is sufficiently large, then $H^s$ embeds into other function spaces, in particular those of continuous or continuously differentiable functions. Assertions of this type are referred to as *Sobolev embedding* type results. The simplest form this takes is the following:

**Proposition 1.4.** *Let $s > \frac{n}{2}$. Then if $f \in \mathcal{S}(\mathbb{R}^n)$ we have the inequality*

$$\big\|f\big\|_{L^\infty(\mathbb{R}^n)} \leq C_{s,n} \cdot \big\|f\big\|_{H^s}$$

*In particular, by density of $\mathcal{S}(\mathbb{R}^n)$ inside $H^s(\mathbb{R}^n)$, every function $f \in H^s(\mathbb{R}^n)$ has a representative in $L^\infty(\mathbb{R}^n)$ satisfying*

$$\big\|f\big\|_{L^\infty(\mathbb{R}^n)} \leq C_{s,n} \cdot \big\|f\big\|_{H^s}.$$

*Proof.* Use the Fourier inversion theorem to write for $f \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned} f(x) &= (2\pi)^{-n} \cdot \int_{\mathbb{R}^n} \widehat{f}(\xi) \cdot e^{ix\cdot\xi}\, d\xi \\ &= (2\pi)^{-n} \cdot \int_{\mathbb{R}^n} (1+|\xi|^2)^{-\frac{s}{2}} \cdot (1+|\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi) \cdot e^{ix\cdot\xi}\, d\xi \end{aligned}$$

Then use the Cauchy-Schwarz inequality to infer that

$$\begin{aligned} &\Big|(2\pi)^{-n} \cdot \int_{\mathbb{R}^n} (1+|\xi|^2)^{-\frac{s}{2}} \cdot (1+|\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi) \cdot e^{ix\cdot\xi}\, d\xi\Big| \\ &\leq (2\pi)^{-n} \cdot \big\|(1+|\xi|^2)^{-\frac{s}{2}}\big\|_{L^2(\mathbb{R}^n)} \cdot \big\|(1+|\xi|^2)^{\frac{s}{2}} \widehat{f}\big\|_{L^2(\mathbb{R}^n)} \\ &= C_{n,s} \cdot \big\|(1+|\xi|^2)^{\frac{s}{2}} \widehat{f}\big\|_{L^2(\mathbb{R}^n)}, \end{aligned}$$

where we set

$$C_{n,s} := (2\pi)^{-n} \cdot \big\|(1+|\xi|^2)^{-\frac{s}{2}}\big\|_{L^2(\mathbb{R}^n)} < \infty$$

since $s > \frac{n}{2}$. □

We now intend to both generalize and sharpen the preceding proposition. A key technical method for this consists in *frequency localization.*

## 2. Frequency localization and the basics of Littlewood-Paley theory

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ a non-negative function which is supported on the annulus $\frac{1}{2} \leq |x| \leq 4$ (where the norm is the usual Euclidean one), and we furthermore have that

$$\chi(x) = 1,\ 1 \leq |x| \leq 2.$$

Then observe that the function

$$\eta(x) := \sum_{l\in\mathbb{Z}} \chi(\frac{x}{2^l})$$

is in fact $C^\infty(\mathbb{R}^n\backslash\{0\})$, and we have the bounds

$$C_1 \geq \eta(x) \geq 1,\ x \in \mathbb{R}^n\backslash\{0\},$$

for some constant $C_1 > 1$, as well as

$$\big|\nabla\eta(x)\big| \leq \frac{C_2}{|x|},\ x \in \mathbb{R}^n\backslash\{0\},$$

for another constant $C_2$. If we now introduce the cutoffs

$$\psi_l(x) := \frac{\chi(\frac{x}{2^l})}{\eta(x)},$$

then for each $l \in \mathbb{Z}$ we clearly have $\psi_l(x) \in C_0^\infty(\mathbb{R}^n)$ and $\psi_l(x) \neq 0$ only if $|x| \in [\frac{2^l}{2}, 4\cdot 2^l]$. Furthermore, we have that $\psi_l(x)$ is non-negative and bounded from above by a constant independent of $l$, and finally, we have

$$\sum_l \psi_l(x) = 1\, \forall x \in \mathbb{R}^n\backslash\{0\}.$$

We also observe the bounds

$$\big|\nabla_x^k \psi_l(x)\big| \leq C_k \cdot |x|^{-k},\ x \in \mathbb{R}^n,$$

where $C_k$ is independent of $l$.

We can now introduce a *Littlewood-Paley decomposition* of a function $f \in \mathcal{S}(\mathbb{R}^n)$ as follows:

**Definition 2.1.** *Define the frequency localized pieces $P_l f(x)$, $l \in \mathbb{Z}$, as follows:*

$$P_l f(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} \psi_l(\xi) \cdot e^{ix\cdot\xi} \widehat{f}(\xi)\, d\xi.$$

We then observe the following basic

**Lemma 2.2.** *We have the decomposition*

$$f = \sum_{l\in\mathbb{Z}} P_l f.$$

*In fact, the sum converges absolutely and uniformly in $x$.*

*Proof.* Observe that if $\widehat{f} \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$ then we have

$$\widehat{f} = \sum_{l\in\mathbb{Z}} \psi_l(\xi)\widehat{f}$$

and in fact the sum on the right is only a finite sum. In particular, we can write

$$f = \sum_{l\in\mathbb{Z}} P_l f,$$

since integration and summation in the Fourier inversion formula can be interchanged. For general $f \in \mathcal{S}(\mathbb{R}^n)$, set for $\varepsilon > 0$,

$$\widehat{g}_\varepsilon := \widetilde{\chi}_{\varepsilon<\cdot<\varepsilon^{-1}}(\xi) \cdot \widehat{f}(\xi)$$

where $\widetilde{\chi}_{\varepsilon<\cdot<\varepsilon^{-1}} \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$ takes values in $[0, 1]$ and equals 1 on the set $\{|\xi| \in [\varepsilon, \varepsilon^{-1}]\}$. Then write

$$\begin{aligned}\sum_l P_l f &= \sum_l P_l g_\varepsilon + \sum_l P_l(f - g_\varepsilon)\\ &= g_\varepsilon + \sum_l P_l(f - g_\varepsilon),\end{aligned}$$

and observe that

$$\lim_{\varepsilon\to 0} \|f - g_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \le \lim_{\varepsilon\to 0} \|\widehat{f} - \widehat{g}_\varepsilon\|_{L^1(\mathbb{R}^n)} = 0,$$

and similarly

$$\begin{aligned}\lim_{\varepsilon\to 0} \big\|\sum_l P_l(f - g_\varepsilon)\big\|_{L^\infty(\mathbb{R}^n)} &\le \lim_{\varepsilon\to 0} \sum_l \|\psi_l(\widehat{f} - \widehat{g}_\varepsilon)\|_{L^1(\mathbb{R}^n)}\\ &\le C \lim_{\varepsilon\to 0} \|\widehat{f} - \widehat{g}_\varepsilon\|_{L^1(\mathbb{R}^n)}\\ &= 0.\end{aligned}$$

□

Let us now use the localization operators $P_l$ to sharpen and generalize the preceding proposition. To begin with, note that due to the fact that

$$\widehat{fg} = \widehat{f} * \widehat{g},$$

we infer

$$P_0 f(x) = (\check{\psi}_0 * f)(x) = \int_{\mathbb{R}^n} \check{\psi}_0(x - y) \cdot f(y)\, dy.$$

where the function $\check{\psi}_0 \in \mathcal{S}(\mathbb{R}^n)$. We can then immediately deduce that

$$\big|P_0 f(x)\big| \le \big\|\check{\psi}_0\big\|_{L^2(\mathbb{R}^n)} \cdot \big\|f\big\|_{L^2(\mathbb{R}^n)} = C_1 \cdot \big\|f\big\|_{L^2(\mathbb{R}^n)},\ \forall x \in \mathbb{R}^n \tag{2.1}$$

by applying the Cauchy-Schwarz inequality, and this can be interpreted as a *frequency localized version* of Proposition 1.3.

By writing

$$\int_{\mathbb{R}^n} \check{\psi}_0(x - y) \cdot f(y)\, dy = \int_{\mathbb{R}^n} \check{\psi}_0(y) \cdot f(x - y)\, dy$$

and applying Minkowski's integral inequality instead, we also infer the estimate

$$\big\|P_0 f\big\|_{L^2(\mathbb{R}^n)} \le \big\|\check{\psi}_0\big\|_{L^1(\mathbb{R}^n)} \cdot \big\|f\big\|_{L^2(\mathbb{R}^n)} = C_2 \cdot \big\|f\big\|_{L^2(\mathbb{R}^n)}. \tag{2.2}$$

Using a simple interpolation argument, we can also deduce bounds for the norms

$$\|P_0 f\|_{L^p(\mathbb{R}^n)}, 2 < p < \infty,$$

using

**Lemma 2.3.** *For a function $g \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we have the bound*[1]

$$\big\|g\big\|_{L^p(\mathbb{R}^n)} \le \big\|g\big\|_{L^\infty(\mathbb{R}^n)}^{\frac{p-2}{p}} \cdot \big\|g\big\|_{L^2(\mathbb{R}^n)}^{\frac{2}{p}}, \ 2 \le p \le \infty.$$

*Proof.* Write

$$\begin{aligned}\big\|g\big\|_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} |g|^p \, dx = \int_{\mathbb{R}^n} |g|^2 \cdot |g|^{p-2} \, dx \le \big\|g\big\|_{L^\infty(\mathbb{R}^n)}^{p-2} \cdot \int_{\mathbb{R}^n} |g|^2 \, dx \\ = \big\|g\big\|_{L^\infty(\mathbb{R}^n)}^{p-2} \cdot \big\|g\big\|_{L^2(\mathbb{R}^n)}^2,\end{aligned}$$

from which the desired estimate follows by taking the $p$-th root of both sides. □

Using the preceding estimates for $\|P_0 f\|_{L^2(\mathbb{R}^n)}, \|P_0 f\|_{L^\infty(\mathbb{R}^n)}$, we can now infer

$$\big\|P_0 f\big\|_{L^p(\mathbb{R}^n)} \le C_3 \big\|f\big\|_{L^2(\mathbb{R}^n)}, \ p \in [2, \infty], \tag{2.3}$$

for some universal constant $C_3$.

We now replicate this reasoning for more general (logarithmic) frequency $l \in \mathbb{Z}$. Thus write

$$P_l f(x) = (\check{\psi}_l * f)(x) = \int_{\mathbb{R}^n} \check{\psi}_l(x-y) \cdot f(y) \, dy.$$

The fact that $\eta(x) = \eta(2^a x)$ for any $a \in \mathbb{Z}$ implies that

$$\psi_l(x) = \Big(\frac{\chi(\cdot)}{\eta(\cdot)}\Big)\Big(\frac{x}{2^l}\Big) = \psi_0\Big(\frac{x}{2^l}\Big).$$

This implies

$$\check{\psi}_l(x) = 2^{nl} \cdot \check{\psi}_0(2^l x).$$

Then we verify directly that

$$\big\|\check{\psi}_l\big\|_{L^1(\mathbb{R}^n)} = \big\|\check{\psi}_0\big\|_{L^1(\mathbb{R}^n)}, \ \big\|\check{\psi}_l\big\|_{L^2(\mathbb{R}^n)} = 2^{\frac{nl}{2}} \cdot \big\|\check{\psi}_0\big\|_{L^2(\mathbb{R}^n)}.$$

Using Lemma 2.3 as before, we then infer that

$$\big\|P_l f\big\|_{L^p(\mathbb{R}^n)} \le C_3 \cdot 2^{\frac{nl}{2} \cdot \frac{p-2}{p}} \cdot \big\|P_l f\big\|_{L^2(\mathbb{R}^n)}, \ 2 \le p \le \infty. \tag{2.4}$$

We can reformulate this inequality in terms of Sobolev norms as follows:

**Proposition 2.4.** *There is a universal constant $C_n$ such that letting $f \in \mathcal{S}(\mathbb{R}^n)$, $l \in \mathbb{Z}$, we have for $2 \le p \le \infty$*

$$\big\|P_l f\big\|_{L^p(\mathbb{R}^n)} \le C_n \cdot \big\|f\big\|_{\dot{H}^s},$$

*where*

$$s = \frac{n}{2} \cdot \frac{p-2}{p}.$$

[1]This lemma holds for any measure space, not just $\mathbb{R}^n$

*Proof.* This is a consequence of the fact that

$$\begin{aligned}2^{\frac{nl}{2}\cdot\frac{p-2}{p}}\cdot\big\|P_l f\big\|_{L^2(\mathbb{R}^n)} &= 2^{\frac{nl}{2}\cdot\frac{p-2}{p}}\cdot\big\|\psi_l(\xi)\cdot\widehat{f}\big\|_{L^2(\mathbb{R}^n)}\\ &\leq C_5\big\||\xi|^{\frac{n}{2}\cdot\frac{p-2}{p}}\psi_l(\xi)\cdot\widehat{f}\big\|_{L^2(\mathbb{R}^n)}\\ &\leq C\big\||\xi|^{\frac{n}{2}\cdot\frac{p-2}{p}}\cdot\widehat{f}\big\|_{L^2(\mathbb{R}^n)}\\ &= C\big\|f\big\|_{\dot{H}^s},\end{aligned}$$

where in the first equality we have used Plancherel's theorem. □

## 3. The fundamental theorem of Littlewood-Paley theory and Sobolev embedding

We now would like to derive an analogue of Prop. 2.4 without the Fourier operator $P_l$ localizing to frequency $\xi\sim 2^l$. Using Lemma 2.2, we have for $f\in\mathcal{S}(\mathbb{R}^n)$ the simple inequality

$$\big\|f\big\|_{L^p(\mathbb{R}^n)}\leq\sum_{l\in\mathbb{Z}}\big\|P_l f\big\|_{L^p(\mathbb{R}^n)},$$

and for each $l\in\mathbb{Z}$ we can invoke Prop. 2.4. However, the sum

$$\sum_{l\in\mathbb{Z}}\big\|P_l f\big\|_{\dot{H}^s}$$

is not majorized by $\big\|f\big\|_{\dot{H}^s}$, for this we would have to replace the $l^1$-sum by a *square-sum*. The fact that this can be accomplished is non-trivial and related to the fundamental theorem of Littlewood-Paley theory. Here we derive the latter assuming some background from Harmonic Analysis (Calderon-Zygmund operators, Mihlin theorem) which are covered in my course on Harmonic Analysis, and further the remarkable *Khinchine's inequality.*

To begin with, we introduce the *Rademacher functions* on $\Omega:=[0,1]\subset\mathbb{R}$, where the notation indicates that we interpret the unit interval as a probability space, and the Rademacher functions as independent identically distributed Random variables. Set

$$b_1(\omega)=+1,\,\omega\in[0,\frac{1}{2}],\,b_1(\omega)=-1,\omega\in(\frac{1}{2},1],$$

and more generally let

$$b_l(\omega)=b_1(2^{l-1}\omega),\omega\in[0,1],$$

where $2^{l-1}\omega$ is interpreted mod 1. Then it it straightforward to verify that

$$\int_\Omega b_l(\omega)b_{l'}(\omega)\,d\omega=\delta_{l,l'},$$

and the $\{b_l\}$ form indeed independent random variables on $\Omega$. We can now state

**Theorem 3.1.** *(Khintchine's inequality) For $1<p<\infty$, there exist positive constants $A_p,B_p$ such that for any $n$-tuple of complex numbers $\{c_l\}_{l=1}^n$, we have*

$$A_p\cdot\big(\sum_l|c_l|^2\big)^{\frac{p}{2}}\leq\int_\Omega\big|\sum_l b_l(\omega)c_l\big|^p\,d\omega\leq B_p\cdot\big(\sum_l|c_l|^2\big)^{\frac{p}{2}}.$$

*Proof.* By considering the real and imaginary parts of the $c_l$ separately, we can reduce to real valued $c_l$. We first prove the right hand inequality, and the left hand one will follow by duality. The main step consists in leveraging the independence of the $\{b_l\}$ in order to derive a powerful bound on the measure of the set $\mathcal{A}_\lambda$ of $\omega\in\Omega$ for which

$$\big|\sum_l b_l(\omega)c_l\big|\geq\lambda>0.$$

The trick is to observe that for any positive real constant $\mu$ we have

$$\int_\Omega e^{\pm\mu\sum_l b_l(\omega)c_l}\,d\omega = \prod_l \int_\Omega e^{\pm\mu b_l(\omega)c_l}\,d\omega = \prod_l \frac{e^{\mu c_l}+e^{-\mu c_l}}{2}.$$

Furthermore, we can bound

$$\frac{e^{\mu c_l}+e^{-\mu c_l}}{2} \leq e^{\frac{\mu^2 c_l^2}{2}}.$$

We conclude that

$$\big|\mathcal{A}_\lambda\big| \leq 2e^{-\mu\lambda}\cdot\int_\Omega e^{\mu\sum_l b_l(\omega)c_l}\,d\omega \leq 2e^{-\mu\lambda}\cdot\prod_l e^{\frac{\mu^2 c_l^2}{2}}.$$

Since this is true for any $\mu$, we can choose $\mu = \frac{\lambda}{\sum_l |c_l|^2}$, which results in

$$\big|\mathcal{A}_\lambda\big| \leq 2e^{-\frac{\lambda^2}{2\sum_l |c_l|^2}}.$$

The upper bound in Khintchine's inequality is now straightforward to obtain: use that

$$\begin{aligned}\int_\Omega \big|\sum_l b_l(\omega)c_l\big|^p\,d\omega &= \int_0^\infty p\lambda^{p-1}\big|\mathcal{A}_\lambda\big|\,d\lambda\\ &\leq 2\int_0^\infty p\lambda^{p-1}\cdot e^{-\frac{\lambda^2}{2\sum_l |c_l|^2}}\,d\lambda\\ &= B_p\cdot\big(\sum_l |c_l|^2\big)^{\frac{p}{2}}.\end{aligned}$$

To derive the lower bound of Khintchine's inequality, we use that

$$\begin{aligned}\sum_l |c_l|^2 &= \int_\Omega \big|\sum_l b_l(\omega)c_l\big|^2\,d\omega\\ &\leq \big(\int_\Omega \big|\sum_l b_l(\omega)c_l\big|^p\,d\omega\big)^{\frac{1}{p}}\cdot\big(\int_\Omega \big|\sum_l b_l(\omega)c_l\big|^q\,d\omega\big)^{\frac{1}{q}},\end{aligned}$$

where $p, q \in (1,\infty)$ are chosen to be *Holder dual*, i. e.

$$\frac{1}{p}+\frac{1}{q} = 1,$$

and we have used Holder's inequality in the last step. Using the upper bound already proven, we infer that

$$\sum_l |c_l|^2 \leq B_p^{\frac{1}{p}}\cdot\big(\sum_l |c_l|^2\big)^{\frac{1}{2}}\cdot\big(\int_\Omega \big|\sum_l b_l(\omega)c_l\big|^q\,d\omega\big)^{\frac{1}{q}}.$$

The lower bound follows (for $p$ replaced by $q$) with

$$A_q = B_p^{-\frac{q}{p}}.$$

□

The preceding theorem is the main ingredient in the proof of the following fundamental

**Theorem 3.2.** *Let $1 < p < \infty$. Then there exist positive constants $D_p, E_p$ such that the we have*

$$D_p\big\|f\big\|_{L^p(\mathbb{R}^n)} \leq \big\|\big(\sum_l |P_l f|^2\big)^{\frac{1}{2}}\big\|_{L^p(\mathbb{R}^n)} \leq E_p\big\|f\big\|_{L^p(\mathbb{R}^n)}.$$

*In words, the $L^p$-norm of $f$ is comparable to the $L^p$-norm of the Littlewood-Paley square function*

$$\big(\sum_l |P_l f|^2\big)^{\frac{1}{2}}.$$

**Corollary 3.3.** *Assume $\infty > p \geq 2$. Then we have the inequality*

$$\|f\|_{L^p(\mathbb{R}^n)} \leq F_p \cdot \big(\sum_l \|P_l f\|_{L^p(\mathbb{R}^n)}^2\big)^{\frac{1}{2}}.$$

*Proof.* (Cor.) From the first inequality of the preceding theorem, we infer that

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}^n)} &\leq D_p^{-1}\|(\sum_l |P_l f|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^n)}\\ &= D_p^{-1}\|\sum_l |P_l f|^2\|_{L^{\frac{p}{2}}(\mathbb{R}^n)}^{\frac{1}{2}}\\ &\leq D_p^{-1}\big(\sum_l \|P_l f\|_{L^p(\mathbb{R}^n)}^2\big)^{\frac{1}{2}},\end{aligned}$$

where we have taken advantage of Minkowski's inequality for the last step. □

The proof of Theorem 3.2 uses *Mihlin's theorem* as a key ingredient, and we shall use this as a black box:

**Theorem 3.4.** *Let $m(\cdot) \in C^\infty(\mathbb{R}^n\backslash\{0\})$ satisfying the bounds*

$$\big|\nabla_\xi^k m(\xi)\big| \leq C_k \cdot |\xi|^{-k},\, k \geq 0.$$

*Then defining $Tf$ by means of the Fourier transform*

$$\widehat{Tf}(\xi) := m(\xi) \cdot \widehat{f}(\xi)$$

*for $f \in \mathcal{S}(\mathbb{R}^n)$, we have that*

$$\big\|Tf\big\|_{L^p(\mathbb{R}^n)} \leq M_p \cdot \big\|f\big\|_{L^p(\mathbb{R}^n)}$$

*for a suitable finite constant $M_p$, provided $1 < p < \infty$.*

*Proof.* (Theorem 3.2 ) We first prove the upper bound. Letting $\{b_l\}$ be the Rademacher functions from before, we have[2] for $1 < p < \infty$

$$A_p \cdot \big(\sum_l \big|P_l f\big|^2\big)^{\frac{p}{2}} \leq \int_\Omega \big|\sum_l P_l f \cdot b_l(\omega)\big|^p\, d\omega \leq B_p \cdot \big(\sum_l \big|P_l f\big|^2\big)^{\frac{p}{2}}. \tag{3.1}$$

where the inequality holds uniformly in $x \in \mathbb{R}^n$ (i. e. we fix the argument $x$ of $P_l f$). But then using that $\Omega$ is a probability space we have from Holder's inequality that

$$\begin{aligned}\int_\Omega \|\sum_l P_l f \cdot b_l(\omega)\|_{L^p(\mathbb{R}^n)}\, d\omega &\leq \big(\int_\Omega \|\sum_l P_l f \cdot b_l(\omega)\|_{L^p(\mathbb{R}^n)}^p\, d\omega\big)^{\frac{1}{p}}\\ &= \Big\|\big(\int_\Omega \big|\sum_l P_l f \cdot b_l(\omega)\big|^p\, d\omega\big)^{\frac{1}{p}}\Big\|_{L^p(\mathbb{R}^n)}\\ &\leq B_p^{\frac{1}{p}} \cdot \|\big(\sum_l \big|P_l f\big|^2\big)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

We need to recover the upper bound for $\|f\|_{L^p(\mathbb{R}^n)}$ from this. Now the trick is to write (recall Lemma 2.2)

$$f = \int_\Omega \big(\sum_l \widetilde{P}_l g_\omega \cdot b_l(\omega)\big)\, d\omega, \tag{3.2}$$

where we define

$$g_\omega := \sum_{l'} P_{l'} f \cdot b_{l'}(\omega),\, \widehat{\widetilde{P}_l g} = \frac{\psi_l(\xi)}{\sum_l \psi_l^2(\xi)} \cdot \widehat{g}(\xi).$$

[2] See the addendum at the end for a justification of the passage to infinite sums

To check (3.2), use that

$$\int_\Omega \big(\sum_l \widetilde{P}_l g_\omega \cdot b_l(\omega)\big)\, d\omega = \int_\Omega \sum_{l,l'} P_{l'} \widetilde{P}_l f \cdot b_l(\omega) b_{l'}(\omega)\, d\omega$$
$$= \sum_l P_l \widetilde{P}_l f = f.$$

Using Mihlin's theorem and a direct verification, one checks that the operator $\sum_l b_l(\omega)\widetilde{P}_l$ acts boundedly on $L^p(\mathbb{R}^n)$ uniformly in $\omega \in \Omega$. It follows that

$$\|f\|_{L^p(\mathbb{R}^n)} \leq H_p \cdot \int_\Omega \|g_\omega\|_{L^p(\mathbb{R}^n)}\, d\omega \leq H_p \cdot B_p^{\frac{1}{p}} \cdot \big\|\big(\sum_l |P_l f|^2\big)^{\frac{1}{2}}\big\|_{L^p(\mathbb{R}^n)},$$

giving the desired upper bound.

To get the lower bound, use (3.1) to deduce

$$\big(\sum_l |P_l f|^2\big)^{\frac{p}{2}} \leq A_p^{-1} \cdot \int_\Omega \big|\sum_l P_l f b_l(\omega)\big|^p\, d\omega$$

pointwise. Applying $\int_{\mathbb{R}^n}$ and again taking advantage of Theorem 3.4 we obtain

$$\int_{\mathbb{R}^n} \big(\sum_l |P_l f|^2\big)^{\frac{p}{2}}\, dx \leq A_p^{-1} \cdot \int_{\mathbb{R}^n}\int_\Omega \big|\sum_l P_l f b_l(\omega)\big|^p\, d\omega dx$$
$$= A_p^{-1} \cdot \int_\Omega \int_{\mathbb{R}^n} \big|\sum_l P_l f b_l(\omega)\big|^p\, dx d\omega$$
$$\leq D_{p,n} \cdot \|f\|_{L^p(\mathbb{R}^n)}^p$$

□

As a consequence, we can now prove the desired generalization of Proposition 2.4 :

**Proposition 3.5.** *Let $2 \leq p < \infty$, and set*

$$s := \frac{n}{2} \cdot \frac{p-2}{p}.$$

*Then we have for $f \in \mathcal{S}(\mathbb{R}^n)$*

$$\|f\|_{L^p(\mathbb{R}^n)} \leq C_{p,n} \cdot \|f\|_{\dot{H}^s(\mathbb{R}^n)}.$$

*Proof.* Due to Corollary 3.3 , we have

$$\|f\|_{L^p(\mathbb{R}^n)} \leq D_{p,n} \cdot \big(\sum_l \|P_l f\|_{L^p(\mathbb{R}^n)}^2\big)^{\frac{1}{2}}.$$

Thanks to Prop. 2.4, we can bound the term on the left by

$$\big(\sum_l \|P_l f\|_{L^p(\mathbb{R}^n)}^2\big)^{\frac{1}{2}} \leq E_{p,n} \cdot \big(\sum_l \|P_l f\|_{\dot{H}^s(\mathbb{R}^n)}^2\big)^{\frac{1}{2}}$$
$$\leq C_{p,n} \cdot \|f\|_{\dot{H}^s(\mathbb{R}^n)},$$

as desired.

□

## 4. Addendum: justification of (3.1)

We obtain this inequality as limiting version of the case with finitely many summands. Letting $f \in \mathcal{S}(\mathbb{R}^n)$, introduce for $N_{1,2} \in \mathbb{N}$ the frequency-truncated function

$$f_{[-N_1,N_2]} := \sum_{l=-N_1}^{N_2} P_l f.$$

Then we indeed have

$$A_p \cdot \big(\sum_l \big|P_l f_{[-N_1,N_2]}\big|^2\big)^{\frac{p}{2}} \leq \int_\Omega \big|\sum_l P_l f_{[-N_1,N_2]} \cdot b_l(\omega)\big|^p \, d\omega \leq B_p \cdot \big(\sum_l \big|P_l f_{[-N_1,N_2]}\big|^2\big)^{\frac{p}{2}}$$

by direct application of Khintchine's inequality, since the sums are finite. By the earlier argument we then infer that

$$D_p \big\|f_{[-N_1,N_2]}\big\|_{L^p(\mathbb{R}^n)} \leq \big\|(\sum_l |P_l f_{[-N_1,N_2]}|^2)^{\frac{1}{2}}\big\|_{L^p(\mathbb{R}^n)} \leq E_p \big\|f_{[-N_1,N_2]}\big\|_{L^p(\mathbb{R}^n)}$$

The idea then is to let $N_{1,2} \to +\infty$, and to use that

$$\lim_{N_1\to\infty, N_2\to\infty} \big\|f_{[-N_1,N_2]}\big\|_{L^p(\mathbb{R}^n)} = \big\|f\big\|_{L^p(\mathbb{R}^n)},$$
$$\lim_{N_1\to\infty, N_2\to\infty} \big\|(\sum_l |P_l f_{[-N_1,N_2]}|^2)^{\frac{1}{2}}\big\|_{L^p(\mathbb{R}^n)} = \big\|(\sum_l |P_l f|^2)^{\frac{1}{2}}\big\|_{L^p(\mathbb{R}^n)}$$

for $1 < p < \infty$. We show the second limiting relation, leaving the first as an exercise. Observe that

$$\begin{aligned}
&(\sum_l |P_l f_{[-N_1,N_2]}|^2)^{\frac{1}{2}} - (\sum_l \big|P_l f\big|^2)^{\frac{1}{2}} \\
&= \frac{\sum_{l\in[-N_1,N_2]^c} |P_l f|^2}{(\sum_l |P_l f_{[-N_1,N_2]}|^2)^{\frac{1}{2}} + (\sum_l \big|P_l f\big|^2)^{\frac{1}{2}}} \\
&= \big(\sum_{l\in[-N_1,N_2]^c} |P_l f|^2\big)^{\frac{1}{2}} \cdot \frac{\big(\sum_{l\in[-N_1,N_2]^c} |P_l f|^2\big)^{\frac{1}{2}}}{(\sum_l |P_l f_{[-N_1,N_2]}|^2)^{\frac{1}{2}} + (\sum_l \big|P_l f\big|^2)^{\frac{1}{2}}}
\end{aligned}$$

provided $\sum_l \big|P_l f\big|^2 \neq 0$. Then note that since

$$\big(\sum_{l\in[-N_1,N_2]^c} |P_l f|^2\big)^{\frac{1}{2}} \leq \sum_{l\in[-N_1,N_2]^c} |P_l f|,$$

and furthermore

$$0 \leq \frac{\big(\sum_{l\in[-N_1,N_2]^c} |P_l f|^2\big)^{\frac{1}{2}}}{(\sum_l |P_l f_{[-N_1,N_2]}|^2)^{\frac{1}{2}} + (\sum_l \big|P_l f\big|^2)^{\frac{1}{2}}} \leq 1$$

on the set of points where $\sum_l \big|P_l f\big|^2 \neq 0$, it suffices to show that

$$\lim_{N_{1,2}\to+\infty} \big\|\sum_{l\in[-N_1,N_2]^c} |P_l f|\big\|_{L^p(\mathbb{R}^n)} = 0$$

provided $1 < p < \infty$. To see this, observe that (exercise!)

$$\big\|P_l f\big\|_{L^\infty} \leq C \cdot 2^{nl} \cdot \big\|f\big\|_{L^1(\mathbb{R}^n)}, \ \big\|P_l f\big\|_{L^1} \leq C \cdot \big\|f\big\|_{L^1(\mathbb{R}^n)}.$$

Using an elementary interpolation argument as before (exercise), we infer $(1 \leq p \leq \infty)$.

$$\big\|P_l f\big\|_{L^p} \leq C \cdot 2^{\frac{p-1}{p}\cdot nl} \cdot \big\|f\big\|_{L^1(\mathbb{R}^n)}$$

But then we have

$$\big\|\sum_{l<-N_1}|P_l f|\big\|_{L^p(\mathbb{R}^n)}\leq\sum_{l<-N_1}\big\|P_l f\big\|_{L^p(\mathbb{R}^n)}\leq C\cdot 2^{-\frac{p-1}{p}\cdot N_1 l}\cdot\big\|f\big\|_{L^1(\mathbb{R}^n)},$$

and the final term on the right obviously converges to 0 as $N_1\to\infty$. We leave the proof that

$$\lim_{N_2\to\infty}\big\|\sum_{l>N_2}|P_l f|\big\|_{L^p(\mathbb{R}^n)}=0$$

as an exercise (use that $P_l f$ decays rapidly with respect to $l$ since the Fourier transform of $f$ is also a Schwartz function).