CONSERVATION LAWS; UNIQUENESS OF SOLUTIONS

1. Conservation laws for the linear models

A remarkable feature of most of the dispersive equations coming from physical applications is that they are characterized by certain *conservation laws*, which are integral expressions over fixed time slices which a priori should be functions of time, but which are actually *time independent*. Some of these conserved quantities, such as the *energy*, give us an a priori control over the solution, which is often of crucial importance to understand the long time behavior of solutions. Moreover, as we shall see in this lecture, the conserved energy (or other analogous quantities) allow us to make uniqueness assertions about solutions.

1.1. **Conservation laws for the linear wave equation.** Consider the linear wave equation

$$u_{tt} - \triangle u = 0,\ u = u(t,x),\ (t,x) \in \mathbb{R}^{1+n}, \tag{1.1}$$

where we assume as in the first lecture that $u \in C^2(\mathbb{R}^{1+n})$. We can write the wave equation in the following form:

$$\mathrm{div}_{t,x}\big(u_t, -\nabla_x u\big) = 0, \tag{1.2}$$

where we set

$$\mathrm{div}_{t,x}(f, g_1, g_2, \ldots g_n) = \partial_t f + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} g_j.$$

If we take the product of the preceding vanishing divergence relation with u_t, we infer

$$\begin{aligned} 0 = u_t \cdot \mathrm{div}_{t,x}\big(u_t, -\nabla_x u\big) &= \partial_t\big(\frac{1}{2}u_t^2\big) - \mathrm{div}_x\big(u_t \cdot \nabla_x u\big) + \partial_t\big(\frac{1}{2}|\nabla_x u|^2\big) \\ &= \partial_t\big(\frac{1}{2}\big|\nabla_{t,x} u\big|^2\big) - \mathrm{div}_x\big(u_t \cdot \nabla_x u\big). \end{aligned} \tag{1.3}$$

We can derive a *global* and a *local* energy conservation law from this relation.

To begin with, assume that $u \in C^2(\mathbb{R}^{1+n})$ is compactly supported on fixed time slices $t = const$. Then we can integrate (1.5) over the space-time slab $[0,T] \times \mathbb{R}^n$, and obtain

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}^n} \big[\partial_t\big(\frac{1}{2}\big|\nabla_{t,x} u\big|^2\big) - \mathrm{div}_x\big(u_t \cdot \nabla_x u\big)\big]\, dxdt \\ &= \int_{\mathbb{R}^n} \frac{1}{2}\big|\nabla_{t,x} u\big|^2\, dx\big|_{t=T} - \int_{\mathbb{R}^n} \frac{1}{2}\big|\nabla_{t,x} u\big|^2\, dx\big|_{t=0}. \end{aligned}$$

Setting

$$E(T) := \int_{\mathbb{R}^n} \frac{1}{2}\big|\nabla_{t,x} u\big|^2\, dx\big|_{t=T}$$

the *energy of* u at time T, we see that the energy is actually independent of the time. This is a *global conservation law*:

Proposition 1.1. *We have*

$$E(T) = E(0)\ \forall T \in \mathbb{R}.$$

We can also deduce more delicate *local conservation laws*. For some $p = (t_0, x_0) \in \mathbb{R}^{1+n}$ with $t_0 > 0$, consider the backward solid light cone K centered at p and given by

$$K = \{(t,x)\,|\, t \le t_0,\ |t - t_0| \ge |x - x_0|\}.$$

Further consider the *truncated solid light cone*

$$K_{0,t_1} = K \cap \big([0,t_1] \times \mathbb{R}^n\big)$$

for $0 \leq t_1 \leq t_0$. Let

$$M_{0,t_1} := \{(t,x) \,|\, 0 \leq t \leq t_1,\, |t - t_0| = |x - x_0|\}$$

the *mantle* of the solid truncated light cone. We can now integrate (1.5) over K_{0,t_1} and apply the divergence theorem. This results in

$$0 = \int_{\partial K_{0,t_1}} \big(\frac{1}{2}|\nabla_{t,x} u|^2, -u_t \cdot \nabla_x u\big) \cdot \overrightarrow{n}\, d\sigma, \tag{1.4}$$

where $\overrightarrow{n}$ denotes the outward pointing perpendicular unit vector on the boundary. Now observe that

$$\overrightarrow{n} = (1, \overrightarrow{0})$$

on $K_{0,t_1} \cap \{t = t_1\}$ while

$$\overrightarrow{n} = (-1, \overrightarrow{0})$$

on $K_{0,t_1} \cap \{t = 0\}$.
Finally, at a point $(t,x) = (t, x_0 + |x - x_0| \cdot \omega)$, $\omega \in S^{n-1}$ on M_{0,t_1}, we have

$$\overrightarrow{n} = \frac{1}{\sqrt{2}} \cdot (1, \omega).$$

The following key computation reveals that the boundary contribution from the mantle to the preceding integral is *non-negative*:

$$\big(\frac{1}{2}|\nabla_{t,x} u|^2, -u_t \cdot \nabla_x u\big) \cdot (1,\omega) = \frac{1}{2}(u_t - \omega \cdot \nabla_x u)^2 + \frac{1}{2}(|\nabla_x u|^2 - (\omega \cdot \nabla_x u)^2) \geq \frac{1}{2}(u_t - \omega \cdot \nabla_x u)^2 \geq 0,$$

since

$$(\omega \cdot \nabla_x u)^2 \leq |\nabla_x u|^2$$

from the Cauchy-Schwarz inequality. We can thus draw the following conclusion from (1.4):

Proposition 1.2. *Using the notation from before, we have the inequality*

$$\Big(\int_{|x-x_0| \leq |t_1 - t_0|} |\nabla_{x,t} u|^2\, dx\Big)\Big|_{t=t_1} \leq \int_{|x-x_0| \leq t_0} |\nabla_{x,t} u|^2\, dx$$

Letting $t_0 \to +\infty$ and $t_1 = T$ this becomes

$$\int_{\mathbb{R}^n} |\nabla_{x,t} u|^2\, dx|_{t=T} \leq \int_{\mathbb{R}^n} |\nabla_{x,t} u|^2\, dx|_{t=0}$$

and by symmetry between the times $t = T, t = 0$ this again recovers the conservation law Prop. 1.1. The preceding derivation furnishes additional information, however.

One immediate application of the preceding proposition concerns uniqueness of solutions of the linear wave equation:

Corollary 1.3. *Let $u_{1,2} \in C^2(\mathbb{R}^{1+n})$ be two solutions of the linear wave equation on $\mathbb{R}^{1+n}$. Assume using the preceding notation that*

$$u_1(0,x) = u_2(0,x),\; u_{1,t}(0,x) = u_{2,t}(0,x),\; \forall |x - x_0| \leq t_0.$$

Then we have

$$u_1(t,x) = u_2(t,x),\; (t,x) \in K_{0,t_0}.$$

Proof. Observe that $u(t,x) := u_1(t,x) - u_2(t,x)$ is also a solution of the free wave equation, and since $\nabla_{t,x}u(0,x) = 0$ for $|x - x_0| \le t_0$, applying Prop. 1.2 for any $t_1 \in [0,t_0]$ implies

$$\nabla_{t,x}u(t_1,x) = 0\,|x - x_0| \le |t_1 - t_0|.$$

But then for any $T \in [0,t_0]$ and x satisfying $|x - x_0| \le |T - t_0|$ we have

$$u(T,x) = u(T,x) - u(0,x) = \int_0^T u_t(t,x)\,dt = \int_0^T 0\,dt = 0.$$

□

We could have deduced this property from the solution formulae in the cases $n = 1,3$, but here we obtain this property in full generality without explicitly solving the wave equation!

Importantly the energy is not the only conserved quantity for the free wave equation. Another such quantity, the *momentum*, is obtained as follows: multiplying (1.2) by u_{x_j}, $j = 1,2,\dots,n$, we find

$$0 = u_{x_j} \cdot \mathrm{div}_{t,x}\big(u_t, -\nabla_x u\big) = \mathrm{div}_{t,x}\big(u_{x_j} \cdot u_t, -u_{x_j} \cdot \nabla_x u\big) - \frac{1}{2}\partial_{x_j}\big(u_t^2 - |\nabla_x u|^2\big) \tag{1.5}$$

If we integrate this relation over the space-time slab $[0,T] \times \mathbb{R}^n$, we find

$$M_j(T) = M_j(0),\ j = 1,2,\dots,n,$$

where we set $M_j(t) = \int_{\mathbb{R}^n} u_{x_j} \cdot u_t\,dx$. This conservation law is somewhat less useful since the quantities M_j are not positive definite, and hence do not furnish an obvious way to control the size of the solution.

Taking advantage of the obvious commutation properties of derivatives on $\mathbb{R}^{1+n}$, we note that if $\Box u = 0$, then so is $\prod_{j=0}^n \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}} u$, where we set $x_0 := t$, and we assume u is sufficiently regular such that the differentiated function is still of class $C^2(\mathbb{R}^{1+n})$. This is certainly the case when $u \in C^\infty(\mathbb{R}^{1+n})$. Under this assumption, we then obtain infinitely many more conserved quantities, namely

$$E\Big(\prod_{j=0}^n \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}} u\Big),\ \big(\alpha_0,\alpha_1,\dots,\alpha_n\big) \in \mathbb{N}^{n+1}. \tag{1.6}$$

What distinguishes the original quantities E, M_j is that for many interesting *nonlinear wave equations*, there are analogues for them, while there are no analogues for the family of conservation laws (1.6).

1.2. **Conservation laws for the linear Schrodinger equation.** By contrast to the linear wave equation, there are *two* natural positive definite conserved quantities for the linear Schrodinger equation, and which are also preserved for a class of important nonlinear Schrodinger equations (NLS). These are the *mass* and the *energy*: letting $\psi(t,x)$ a solution of

$$i\partial_t\psi + \triangle\psi = 0,$$

then we set

$$m(t) = \int_{\mathbb{R}^n} |\psi|^2(t,x)\,dx,\ E(t) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\psi|^2(t,x)\,dx.$$

Here we use the notation

$$|\nabla\psi|^2 = \sum_{j=1}^n |\frac{\partial}{\partial x_j}\psi|^2.$$

That these quantities are indeed conserved follows from

Proposition 1.4. *Assume that $\psi \in C^\infty(\mathbb{R}^n)$ is a solution of the Schrodinger equation such that*

$$\psi(t,\cdot) \in \mathcal{S}(\mathbb{R}^n), \psi_t(t,\cdot) \in \mathcal{S}(\mathbb{R}^n)$$

for all $t \in \mathbb{R}$. Then we have

$$m(t) = m(0),\ E(t) = E(0),\ \forall t \in \mathbb{R}.$$

Proof. Note that if ψ is a solution of the Schrodinger equation, then so is $\frac{\partial}{\partial x_j}\psi$, $j = 1, 2, \ldots, n$. The energy conservation follows hence from the mass conservation. For the latter, we observe that

$$
\begin{aligned}
i\frac{d}{dt}\int_{\mathbb{R}^n} |\psi(t,x)|^2\,dx = 2i\int_{\mathbb{R}^n} \operatorname{Im}\left(i\psi_t\bar{\psi}\right)dx &= -2i\int_{\mathbb{R}^n} \operatorname{Im}\left(\triangle\psi\bar{\psi}\right)dx \\
&= +2i\int_{\mathbb{R}^n} \operatorname{Im}\left(|\nabla\psi|^2\right)dx \\
&= 0.
\end{aligned}
$$

□

Using the preceding proposition, we can immediately infer the following uniqueness result:

Corollary 1.5. *Let $\psi \in C^\infty(\mathbb{R}^n)$ is a solution of the Schrodinger equation such that*

$$\psi(t,\cdot) \in \mathcal{S}(\mathbb{R}^n), \psi_t(t,\cdot) \in \mathcal{S}(\mathbb{R}^n)$$

for all $t \in \mathbb{R}$, and such that $\psi(0,x) = 0$ for all $x \in \mathbb{R}^n$. Then

$$\psi = 0.$$

1.3. **Conservation laws for the linear KdV equation.** Finally we state

Proposition 1.6. *Assume that $\psi \in C^\infty(\mathbb{R}^n)$ is a solution of the linear KdV equation such that*

$$\psi(t,\cdot) \in \mathcal{S}(\mathbb{R}^n), \psi_t(t,\cdot) \in \mathcal{S}(\mathbb{R}^n)$$

for all $t \in \mathbb{R}$. Then setting

$$m(t) := \int_{\mathbb{R}} \psi^2\,dx,$$

we have

$$m(t) = m(0)\,\forall t \in \mathbb{R}.$$

Proof. This is again a consequence of integration by parts. Using that

$$\psi_t + \psi_{xxx} = 0,$$

we find

$$
\begin{aligned}
\frac{d}{dt}m(t) = 2\int_{\mathbb{R}} \psi\psi_t\,dx = -2\int_{\mathbb{R}} \psi\psi_{xxx}\,dx &= +2\int_{\mathbb{R}} \psi_x\psi_{xx}\,dx \\
&= \int_{\mathbb{R}} \frac{d}{dx}\left(\psi_x^2\right)dx \\
&= 0.
\end{aligned}
$$

□

Remark 1.7. It is an amazing fact that the *true nonlinear KdV equation*

$$\psi_t + \psi_{xxx} - 6\psi\psi_x = 0$$

admits an *infinite family* of conservation laws, involving expressions with more and more derivatives. This is related to the fact that this model is *completely integrable*, and hence belongs to a class of very special nonlinear PDE. Most dispersive PDE do not have this structure.

2. A more abstract approach to conservation laws: energy momentum tensor for the wave equation

We follow the presentation in Shatah-Struwe(2000). Let $u \in C^2\left(\mathbb{R}^{n+1}\right)$ a solution of the free wave equation: $\Box u = u_{tt} - \triangle u = 0$. Consider the formal space time integral, called a *Lagrangian action functional*

$$\mathcal{L}(u) := \int_{\mathbb{R}^{n+1}} \left(|u_t|^2 - |\nabla_x u|^2\right) dxdt. \tag{2.1}$$

Let us consider the *formal variation* of this expression with respect to a compactly supported function $\phi \in C_0^2\left(\mathbb{R}^{n+1}\right)$. This means we consider

$$\frac{\partial}{\partial \varepsilon}\mathcal{L}\big(u + \varepsilon\phi\big)\big|_{\varepsilon=0} = \int_{\mathbb{R}^{n+1}} 2\big(u_t \cdot \phi_t - \nabla_x u \cdot \nabla_x \phi\big)\, dxdt.$$

This last integral is now well-defined, and performing integration by parts, this becomes

$$\int_{\mathbb{R}^{n+1}} 2\big(u_t \cdot \phi_t - \nabla_x u \cdot \nabla_x \phi\big)\, dxdt = -2\int_{\mathbb{R}^{n+1}} \Box u \cdot \phi \, dtdx = 0.$$

It follows that u is a *critical point for the Lagrangian action functional* $\mathcal{L}(u)$.

The Lagrangian in turn can be used to derive conservation laws, due to the fact that there is no explicit dependence of t, x in it, which makes it invariant under general coordinate changes. First, let us write the integrand in the Lagrangian in the following succinct way:

$$|u_t|^2 - |\nabla_x u|^2 = \sum_{\alpha,\beta=0}^{n} \eta^{\alpha\beta} \cdot \partial_\alpha u \cdot \partial_\beta u, \tag{2.2}$$

where we define $\partial_0 = \partial_t$, $\partial_i = \partial_{x_i}$, $i = 1, 2, \dots, n$. Also, we have

$$\eta^{\alpha\beta} = \begin{pmatrix} 1 & 0\dots & 0 \\ 0 & -1 & \dots 0 \\ 0 & 0 & \dots -1 \end{pmatrix}$$

Write

$$L(\nabla u, \eta) := \sum_{\alpha,\beta=0}^{n} \eta^{\alpha\beta} \cdot \partial_\alpha u \cdot \partial_\beta u =: \eta^{\alpha\beta} \cdot \partial_\alpha u \cdot \partial_\beta u.$$

Here we omit the summation sign at the end, with the convention that one sums over identical indices occurring twice. The one makes

Definition 2.1. *The family of functions*

$$T_{\alpha\beta} := \frac{1}{2}\eta_{\alpha\beta} L(\nabla u, \eta) - \frac{\partial L(\nabla u, \eta)}{\partial \eta^{\alpha\beta}},\ 0 \le \alpha, \beta \le n,$$

is called the energy momentum tensor. We also set

$$T^{\alpha\beta} = \eta^{\alpha\gamma}\eta^{\beta\delta} T_{\gamma\delta}.$$

For example, we observe that

$$T_{00} = \frac{1}{2} \cdot \left(|u_t|^2 - |\nabla_x u|^2\right) - |u_t|^2 = -\frac{1}{2} \cdot \left(|u_t|^2 + |\nabla_x u|^2\right)$$

is the energy density, up to a constant.

The conservation laws for the wave equation which we observed on an ad hoc basis now emerge as a consequence of the following

Proposition 2.2. *If $u \in C^2(\mathbb{R}^{n+1})$ is a solution of the free wave equation, then we have the following vanishing divergence relations:*

$$\sum_{\alpha=0}^{n} \partial_\alpha T^{\alpha\beta} =: \partial_\alpha T^{\alpha\beta} = 0,\ 0 \leq \beta \leq n.$$

Proof. Instead of direct calculation, we use the observation that the Lagrangian action functional is invariant under general coordinate changes to derive this. Specifically, we shall use the invariance under *one parameter groups of diffeomorphisms.* Let

$$U \subset \mathbb{R}^{n+1}$$

a pre-compact open subset, let $\tau \in C_0^\infty\big(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}\big)$ with support contained inside U, and let $\varepsilon_* > 0$ be small enough such that setting

$$G_\varepsilon := \mathrm{id} + \varepsilon\tau \in C^\infty\big(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}\big),$$

we have that

$$G_\varepsilon\big(U\big) = U,\ 0 \leq \varepsilon < \varepsilon_*,$$

and G_ε is a diffeomorhpism. Then we observe that letting $\big|dG_\varepsilon\big|$ be the determinant of the Jacobian matrix

$$dG_\varepsilon := \big(\frac{\partial G_\varepsilon^\beta}{\partial \alpha}\big)_{0\leq\alpha,\beta\leq n},$$

we have that (exercise)

$$\big|dG_\varepsilon\big| = 1 + \varepsilon \cdot \partial_\alpha \tau^\alpha + O\big(\varepsilon^2\big).$$

Now from the basic change of variables formula we infer that

$$\int_U L(\nabla u, \eta) \circ G_\varepsilon \cdot \big|dG_\varepsilon\big|\, dxdt = \int_{G_\varepsilon(U)} L(\nabla u, \eta)\, dxdt = \int_U L(\nabla u, \eta)\, dxdt$$

It follows that

$$\frac{d}{d\varepsilon}\Big(\int_U L(\nabla u, \eta) \circ G_\varepsilon \cdot \big|dG_\varepsilon\big|\, dxdt\Big)\big|_{\varepsilon=0} = 0. \tag{2.3}$$

We shall explicitly compute this derivative. To begin with, we would like to interpret

$$L(\nabla u, \eta) \circ G_\varepsilon = \eta_\varepsilon^{\alpha\beta} \cdot \partial_\alpha u_\varepsilon \cdot \partial_\beta u_\varepsilon,$$

where we set

$$u_\varepsilon := u \circ G_\varepsilon,$$

and $\eta_\varepsilon^{\alpha\beta}$ is a suitably modified space-time metric.

Observe that letting $\partial_\alpha u = p_\alpha$, $\alpha = 0, \ldots, n$, we have

$$\partial_\alpha u_\varepsilon = \partial_\alpha u + \varepsilon \cdot \partial_\alpha\big(\partial_\beta u \cdot \tau_\beta\big) + O\big(\varepsilon^2\big),$$

In turn we can write

$$\begin{aligned}
&\int_U \eta_\varepsilon^{\alpha\beta} \cdot \partial_\varepsilon\big(\partial_\alpha u_\varepsilon \cdot \partial_\beta u_\varepsilon\big)\big|_{\varepsilon=0}\, dtdx \\
&= 2\int_U \eta_\varepsilon^{\alpha\beta} \cdot \partial_\beta u_\varepsilon \cdot \partial_\alpha\big(\partial_\gamma u \cdot \tau_\gamma\big)\big|_{\varepsilon=0}\, dtdx \\
&= -2\int_U \partial_\alpha\big(\eta_\varepsilon^{\alpha\beta} \cdot \partial_\beta u_\varepsilon\big) \cdot \big(\partial_\gamma u \cdot \tau_\gamma\big)\big|_{\varepsilon=0}\, dtdx \\
&= 0,
\end{aligned}$$

since u solves the free wave equation. It follows that in order to evaluate the left hand side of (2.3), we only need to consider the case when the operator $\frac{d}{d\varepsilon}$ falls on $\eta_\varepsilon^{\alpha\beta}$ or on $\big|dG_\varepsilon\big|$. For the former case, we need to compute the coefficient matrix $\eta_\varepsilon^{\alpha\beta}$ to linear order in ε. Letting G_ε^{-1} the inverse of the diffeomorphism G_ε, we infer that

$$\big(\partial_\beta\big(G_\varepsilon\big)\circ G_\varepsilon^{-1}\big)^\gamma \cdot \partial_\alpha\big(G_\varepsilon^{-1}\big)^\beta = \delta_\alpha^\gamma,$$
$$\partial_\beta\big(G_\varepsilon\big)^\gamma \cdot \partial_\alpha\big(G_\varepsilon^{-1}\big)^\beta \circ G_\varepsilon = \delta_\alpha^\gamma$$

where we sum over β and the right hand side is the usual delta function which vanishes except $\delta_\alpha^\gamma = 1$ when $\alpha = \gamma$. We now deduce that (summing over repeated indices)

$$\eta_\varepsilon^{\alpha\beta} = \eta^{\gamma\delta}\cdot\big[\big(\partial_\gamma\big(G_\varepsilon^{-1}\big)\circ G_\varepsilon\big]^\alpha \cdot \big(\partial_\delta\big(G_\varepsilon^{-1}\big)\circ G_\varepsilon\big]^\beta$$

In fact, in light of $\partial_\alpha u_\varepsilon = \big(\partial_\nu u\circ G_\varepsilon\big)\cdot\partial_\alpha G_\varepsilon^\nu$, we then get

$$\begin{aligned}
&\eta_\varepsilon^{\alpha\beta}\cdot\partial_\alpha u_\varepsilon\cdot\partial_\beta u_\varepsilon\\
&= \eta^{\gamma\delta}\cdot\big[\big(\partial_\gamma\big(G_\varepsilon^{-1}\big)\circ G_\varepsilon\big]^\alpha\cdot\big(\partial_\delta\big(G_\varepsilon^{-1}\big)\circ G_\varepsilon\big]^\beta\cdot\big(\partial_\nu u\circ G_\varepsilon\big)\cdot\partial_\alpha G_\varepsilon^\nu\cdot\big(\partial_\mu u\circ G_\varepsilon\big)\cdot\partial_\beta G_\varepsilon^\mu\\
&= \eta^{\nu\mu}\cdot(\partial_\nu u\circ G_\varepsilon\big)\cdot\big(\partial_\mu u\circ G_\varepsilon\big)\\
&= L(\nabla u,\eta)\circ G_\varepsilon
\end{aligned}$$

We then compute that

$$\begin{aligned}
\eta_\varepsilon^{\alpha\beta} &= \eta^{\gamma\delta}\cdot\big(\delta_\gamma^\alpha - \varepsilon\partial_\gamma\tau^\alpha + O(\varepsilon^2)\big)\cdot\big(\delta_\delta^\beta - \varepsilon\partial_\delta\tau^\beta + O(\varepsilon^2)\big)\\
&= \eta^{\alpha\beta} - \varepsilon\big(\partial^\beta\tau^\alpha + \partial^\alpha\tau^\beta\big) + O\big(\varepsilon^2\big).
\end{aligned}$$

Coming back to (2.3), we see that the left hand side equals

$$\begin{aligned}
&\int_U \Big(L(\nabla u,\eta)\cdot\partial_\alpha\tau^\alpha - \frac{\partial L(\nabla u,\eta)}{\partial\eta^{\alpha\beta}}\cdot\big(\partial^\beta\tau^\alpha + \partial^\alpha\tau^\beta\big)\Big)\,dtdx\\
&= -\int_U \tau^\beta\partial^\alpha\Big(\eta_{\alpha\beta}L(\nabla u,\eta) - 2\frac{\partial L(\nabla u,\eta)}{\partial\eta^{\alpha\beta}}\Big)\,dtdx
\end{aligned}$$

Since τ was arbitrary, the proposition follows.

$\square$