INTRODUCTION TO BASIC DISPERSIVE PDE.

This course is intended to provide an introduction to some basic dispersive PDE and the most fundamental technical tools used to study them. Some of these tools are fairly technically sophisticated and rely on a certain amount of harmonic analysis. We shall gently introduce these tools as we go along, but for now we simply consider some of the key examples that will accompany us throughout this course.

So what is a 'dispersive PDE'? Roughly speaking, the solutions of such partial differential equations, which are all *evolution equations*, i. e. they describe functions $\phi(t,x)$ which depend both on time t and space x, have the tendency to spread out and decay in amplitude as $t \to \pm\infty$. More specifically, they do so since they are superpositions of certain fundamental building blocks, namely travelling waves, whose propagation velocity is *frequency dependent*, and it is this latter effect which causes the solutions to spread out and decay. This will be rendered very explicit by means of the Fourier representation.

Throughout this course, we will work with equations on $\mathbb{R}^{1+n} = \{(t,x) \,|\, t \in \mathbb{R},\, x \in \mathbb{R}^n\}$. One can also study the models introduced below on bounded domains with various boundary conditions, which often dramatically changes the properties of the solutions and the theory required.

In the sequel we consider some fundamental linear dispersive models. Understanding the fine properties of the solutions of these is typically required to approach the much more sophisticated nonlinear models later on. In fact, the most interesting phenomena for the latter arise when there is a delicate balance between the dispersive spreading effect causing decay and nonlinear effects causing growth, sometimes resulting in singularities, sometimes in soliton like phenomena.

1. The linear wave equation

1.1. **Elementary approach to the one and three dimensional cases.** Maybe the oldest partial differential equation, the linear wave equation, describes approximately the vibrations of a vibrating string, and is given by

$$\phi_{tt} - \phi_{xx} = 0,\ \phi = \phi(t,x),\ (t,x) \in \mathbb{R}^{1+1}, \tag{1.1}$$

and more generally

$$\phi_{tt} - \triangle\phi = 0,\ \phi = \phi(t,x),\ (t,x) \in \mathbb{R}^{1+n}. \tag{1.2}$$

In fact, there is an important difference between the one (spatial) dimensional and the $n \geq 2$ (spatial) dimensional case, as the equation *is not of dispersive character* in the former:

Proposition 1.1. *Assume that $\phi \in C^2(\mathbb{R}^{1+1})$ solves (1.1), then there exist functions $g \in C^2(\mathbb{R}), h \in C^2(\mathbb{R})$, such that*

$$\phi(t,x) = g(t+x) + h(t-x).$$

In other words, the solution ϕ is the sum of a travelling wave $h(t-x)$ propagating to the right and a travelling wave $g(t+x)$ propagating to the left, and each one of these preserves its shape (and its amplitude).

Proof. Introduce the variables $u = t+x, v = t-x$, and write

$$\phi(t,x) = \psi(u,v) = \phi(\frac{u+v}{2}, \frac{u-v}{2}),$$

whence $\psi \in C^2(\mathbb{R}^{1+1})$ also. Then we have

$$(\partial_t^2 - \partial_x^2)\big(\psi(u,v)\big) = \big((\partial_u + \partial_v)^2 - (\partial_u - \partial_v)^2\big)\psi = 4\partial_{uv}^2\psi = 0.$$

The last equation implies

$$\partial_v \psi = G(v),\ G \in C^1(\mathbb{R}),$$

and in turn this implies

$$\psi(u,v) = g(v) + h(u),$$

where $g = \int G\, dv \in C^2(\mathbb{R})$, $h \in C^2(\mathbb{R})$. □

As a consequence, we can explicitly solve the *Cauchy problem* for the linear wave equation in one dimension:

Corollary 1.2. *The solution $\phi(t,x) \in C^2(\mathbb{R}^{1+1})$ of*

$$\phi_{tt} - \phi_{xx} = 0,\ \phi(0,x) = \phi_0(x),\ \phi_t(0,x) = \phi_1(x) \tag{1.3}$$

where $\phi_0 \in C^2(\mathbb{R})$, $\phi_1 \in C^1(\mathbb{R})$, is given by

$$\phi(t,x) = \frac{1}{2}(\phi_0(t+x) + \phi_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} \phi_1(s)\, ds$$

Proof. According to the preceding proposition we have $\phi(t,x) = g(x+t) + \widetilde{h}(x-t)$, where $\widetilde{h}(x-t) := h(t-x)$, with $g, \widetilde{h} \in C^2(\mathbb{R})$. The initial conditions give

$$g(x) + \widetilde{h}(x) = \phi_0(x),\ g'(x) - \widetilde{h}'(x) = \phi_1(x),$$

which results in

$$g = \frac{1}{2}(\phi_0 + \int \phi_1(s)\, ds),\ \widetilde{h} = \frac{1}{2}(\phi_0 - \int \phi_1(s)\, ds).$$

Here $g, \widetilde{h}$ are determined up to a constant $\pm a$. Inserting these in the formula for $\phi(t,x)$ and re-arranging terms the corollary follows. □

Still continuing in this elementary spirit we now consider the linear wave equation in $n = 3$ spatial dimensions, where the equation is indeed dispersive in character.

Proposition 1.3. *Assume that $\phi \in C^2(\mathbb{R}^{1+3})$ satisfies the linear wave equation*

$$\phi_{tt} - \triangle\phi = 0,\ \phi(0,x) = \phi_0(x),\ \phi_t(0,x) = \phi_1(x) \tag{1.4}$$

with $\phi_0 \in C^2(\mathbb{R}^3), \phi_1 \in C^1(\mathbb{R}^3)$. Then $\phi(t,x)$ is given by the Kirchhoff formula

$$\phi(t,x) = \frac{1}{4\pi t}\int_{|x-y|=|t|} \phi_1(y)\, d\sigma_y + \partial_t\big(\frac{1}{4\pi t}\int_{|x-y|=|t|} \phi_0(y)\, d\sigma_y\big)$$

where $d\sigma_y$ denotes the standard surface measure on the sphere $|x-y| = |t|$.

Corollary 1.4. *Assuming $\phi_0 \in C_0^2, \phi_1 \in C_0^1$, the solution ϕ exhibits dispersive decay:*

$$\big|\phi(t,x)\big| \leq (1+|t|)^{-1} \cdot C(\phi_0, \phi_1).$$

Moreover, and this is specific to the linear wave equation in odd spatial dimensions, we have the sharp Huyghen's principle: assuming that ϕ_0, ϕ_1 are supported in $B_L(0) = \{|x| \leq L\}$, we have

$$\phi(t,x) = 0$$

provided $\big||t| - |x|\big| > L$. In particular, the solutions to the linear wave equation propagate inside the light cone $|x| \leq |t| + L$. If $\phi_0 = 0$ and $\phi_1 \geq 0$, then $\phi(t,x) \geq 0$ for $t \geq 0$.

Proof. Observe that the linear wave equation is *translation invariant*: this means that if $\phi(t,x)$ solves it, then so does $\phi(t+t_*, x+x_*)$ for any point $(t_*, x_*) \in \mathbb{R}^{1+3}$. Setting $t_* = 0$, and varying $x_* = x$, we see that it suffices to prove the formula for $x = 0$. We shall first consider the case $(\phi_0, \phi_1) = (0, \phi_1)$ for simplicity, the general case being handled similarly. Note that the reflection symmetry

$$\phi(t,x) \longrightarrow -\phi(-t,x)$$

carries free waves into free waves, leaves the data $(0, \phi_1)$ at time $t = 0$ invariant, and also leaves the part of the Kirchhoff formula without ϕ_0 invariant. This allows us to reduce to the situation $t \geq 0$. Consider spherical coordinates $(r, \omega) \in \mathbb{R}_+ \times S^2$ centered at the origin $x = 0$ in $\mathbb{R}^3$. We can then express the Laplace operator as

$$\triangle = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\triangle_{S^2}.$$

We now use a trick to reduce the three dimensional wave equation to the one dimensional one studied before. In fact, given $\phi \in C^2(\mathbb{R}^{1+3})$, introduce the auxiliary function

$$\psi(t, r) := r \cdot \int_{S^2} \phi(t, r\omega)\, d\sigma_\omega$$

provided $r > 0$. This function is easily seen to be of class C^2 on $\{(t, r)\,|r > 0\}$, and in fact extending it as an *odd* function to all of $\mathbb{R}$ results in a C^2-function on $\mathbb{R}^{1+1}$ (exercise!). Moreover, *it satisfies the linear wave equation* there:

$$\begin{aligned}(\partial_t^2 - \partial_r^2)\psi(t, r) &= r \cdot \int_{S^2} (\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r)\phi(t, r\omega)\, d\sigma_\omega \\ &= r \cdot \int_{S^2} (\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r - \frac{1}{r^2}\triangle_{S^2})\phi(t, r\omega)\, d\sigma_\omega \\ &= 0,\end{aligned}$$

where in the second step we used that $\int_{S^2} \triangle_{S^2}\big(\phi(t, r\omega)\big)\, d\sigma_\omega = 0$. We can thus apply Corollary 1.2 to infer that

$$\psi(t, r) = \frac{1}{2}\int_{r-t}^{r+t} \psi_1(s)\, ds,\ \psi_1(r) = r \cdot \int_{S^2} \phi_1(|r|\omega)\, d\sigma_\omega,$$

where the formula on the right for ψ_1 means we extend the function $r \cdot \int_{S^2} \phi_1(r\omega)\, d\sigma_\omega$ for $r > 0$ as an odd function to $\mathbb{R}$. Since the function ψ_1 is odd, we can reformulate the expression for ψ and $t > 0, r > 0$, as follows:

$$\psi(t, r) = \frac{1}{2}\int_{r-t}^{r+t} \psi_1(s)\, ds = \frac{1}{2}\int_{|r-t|}^{r+t} \psi_1(s)\, ds.$$

In order to recover $\phi(t, r\omega)$ for $t > 0$ and at $r = 0$, the key observation is that

$$\begin{aligned}\phi(t, 0) &= \frac{1}{4\pi} \cdot \partial_r\big(r \cdot \int_{S^2} \phi(t, r\omega)\, d\sigma_\omega\big)|_{r=0} \\ &= \frac{1}{4\pi} \cdot \partial_r\big(\frac{1}{2}\int_{|r-t|}^{r+t} \psi_1(s)\, ds\big)|_{r=0} \\ &= \frac{1}{4\pi} \cdot t\int_{S^2} \phi_1(t\omega)\, d\sigma_\omega \\ &= \frac{1}{4\pi t} \cdot \int_{|y|=t} \phi_1(y)\, d\sigma_y.\end{aligned}$$

This gives the Kirchhoff formula in case $\phi_0 = 0$. Observe that the preceding argument implies that the solution is unique.

For the general case, one also needs to add the term $\frac{1}{2}[\psi_0(r + t) + \psi_0(r - t)]$ to obtain $\psi(t, r)$, and computing

$$\frac{1}{4\pi} \cdot \partial_r\big(\frac{1}{2}[\psi_0(r + t) + \psi_0(r - t)]\big)\big|_{r=0},\ \psi_0(r) = r \cdot \int_{S^2} \phi_0(|r|\omega)\, d\sigma_\omega$$

results in the second term in the Kirchoff formula.

□

1.2. **Notation.** For later reference, we introduce the *d'Alembertian*

$$\Box = \partial_{tt} - \triangle = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

1.3. **Using the Fourier representation to understand dispersion.** The preceding subsection uses a 'physical space approach' to solving the linear wave equation, with remarkable qualitative and quantitative conclusions in the $n = 3$ (spatial) dimensional case. An alternative approach, which generalizes much more easily to other important situations, takes advantage of the Fourier representation of the solution. In this course, we assume familiarity with the Fourier transform on $\mathbb{R}^n$. Given $f \in L^1(\mathbb{R}^n)$, we can define

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx,$$

and if also $g(\xi) \in L^1(\mathbb{R}^n)$, we have the 'inverse Fourier transform'

$$\check{g}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi)\,d\xi.$$

To ensure that $\widehat{f} \in L^1(\mathbb{R}^n)$, it suffices to require $f \in \mathcal{S}(\mathbb{R}^n)$, the space of Schwartz functions, which is mapped into itself by the Fourier transform.
Assuming for now that the solution $\phi(t, x) \in C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ for any fixed t, we can use the representation

$$\phi(t, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{\phi}(t, \xi) e^{ix\cdot\xi} d\xi,\ \widehat{\phi}(t, \xi) = \int_{\mathbb{R}^n} \phi(t, x) e^{-ix\cdot\xi}\,dx. \tag{1.5}$$

Simple integration by parts leads to the formula

$$\widehat{\triangle\phi}(t, \xi) = \int_{\mathbb{R}^n} \triangle\phi(t, x) e^{-ix\cdot\xi}\,dx = -|\xi|^2 \widehat{\phi}(t, \xi).$$

In terms of the Fourier coefficients $\widehat{\phi}(t, \xi)$, the linear wave equation is then translated into

$$\partial_{tt}\widehat{\phi}(t, \xi) + |\xi|^2 \widehat{\phi}(t, \xi) = 0. \tag{1.6}$$

We interpret this as an ordinary differential equation for each fixed $\xi \in \mathbb{R}^n$, which we can solve by means of

$$\widehat{\phi}(t, \xi) = c_1(\xi) e^{it|\xi|} + c_2(\xi) e^{-it|\xi|}.$$

Imposing initial conditions $\phi(0, x) = f(x)$, $\phi_t(0, x) = g(x)$ at $t = 0$ (again assumed in $C_0^\infty(\mathbb{R}^n)$ for now), we get the conditions

$$c_1(\xi) + c_2(\xi) = \widehat{f}(\xi),\ i|\xi|[c_1(\xi) - c_2(\xi)] = \widehat{g}(\xi).$$

This implies

$$c_1(\xi) = \frac{1}{2}[\widehat{f}(\xi) + \frac{1}{i|\xi|}\widehat{g}(\xi)],\ c_2(\xi) = \frac{1}{2}[\widehat{f}(\xi) - \frac{1}{i|\xi|}\widehat{g}(\xi)],$$

which then results in the following formula for $\phi(t, x)$:

$$\begin{aligned} \phi(t, x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(t|\xi| + x\cdot\xi)} \cdot \frac{1}{2}[\widehat{f}(\xi) + \frac{1}{i|\xi|}\widehat{g}(\xi)]\,d\xi \\ &+ (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(-t|\xi| + x\cdot\xi)} \cdot \frac{1}{2}[\widehat{f}(\xi) - \frac{1}{i|\xi|}\widehat{g}(\xi)]\,d\xi \end{aligned} \tag{1.7}$$

It is easy to see that in the $n = 1$ dimensional case, we can write the preceding in the form

$$\phi(t, x) = h(t - x) + g(t + x),$$

in accordance with Proposition 1.1. We also observe the following interpretation from physics in this case: setting (one calls k the wave number and ω the angular frequency)

$$e^{i(\pm t|\xi| + x\cdot\xi)} = e^{i(kx - \omega t)},\ k = \xi, \omega = \pm|\xi|,$$

then for $k \neq 0$ the quantity

$$\frac{\omega}{k},$$

which is called the *phase velocity*, equals the quantity

$$\frac{d\omega}{dk}$$

which is called the *group velocity*. This is technically equivalent to absence of dispersion for this case, as we have already observed. Below we will encounter one dimensional linear PDEs which are dispersive, i. e. phase velocity and group velocity differ.

On the other hand, as far as the linear wave equation is concerned, for $n \geq 2$, the exponential factors

$$e^{i(\pm t|\xi|+x\cdot\xi)} = e^{i|\xi|(\pm t+x\cdot\frac{\xi}{|\xi|})}$$

correspond to travelling waves propagating in different directions $\frac{\xi}{|\xi|} \in S^{n-1}$, and this causes dispersion, and in particular amplitude decay (as we have seen via a different route in the $n = 3$ case). We will revisit later the issue as to how to deduce amplitude decay for the linear wave equation from the Fourier representation in the case $n \geq 2$.

2. The linear Schrodinger equation

The Schrodinger equation is of more recent vintage and arose at the beginnings of Quantum Mechanics. We will content ourselves with the case of trivial potential, when the equation becomes

$$i\partial_t\psi + \triangle\psi = 0,\ \psi = \psi(t, x),\ i = \sqrt{-1},\ (t, x) \in \mathbb{R}^{1+n}$$

Let us consider the one dimensional case $n = 1$, which is already dispersive. To see this, we use the Fourier representation. Throughout we assume that the solutions are 'nice enough', and specifically $\psi(t, \cdot) \in \mathcal{S}(\mathbb{R})$ for each $t \in \mathbb{R}$, so that we are justified in using the Fourier representation. Furthermore, we assume that also ψ_t is Schwartz with respect to x, and the corresponding Fourier integrals converge uniformly in ξ with respect to t so that the manipulations below (moving the time derivative inside the integral) are all justified. Then write

$$\psi(t, x) = (2\pi)^{-1}\int_{\mathbb{R}}\widehat{\psi}(t, \xi)e^{ix\cdot\xi}\,d\xi,\ \widehat{\psi}(t, \xi) = \int_{\mathbb{R}}\psi(t, x)e^{-ix\cdot\xi}\,dx.$$

Then the Schrodinger equation is translated into

$$i\partial_t\widehat{\psi} - \xi^2\widehat{\psi} = 0,$$

which means that

$$\widehat{\psi}(t, \xi) = e^{-it\xi^2}\cdot c(\xi).$$

Imposing an initial condition $\psi(0, x) = f(x)$ results in $c(\xi) = \widehat{f}(\xi)$, and we obtain the solution formula

$$\psi(t, x) = (2\pi)^{-1}\int_{\mathbb{R}}e^{i(x\cdot\xi - t\xi^2)}\widehat{f}(\xi)\,d\xi.$$

Setting $k = \xi$, $\omega = k^2$, we have

$$\frac{\omega}{k} \neq \frac{d\omega}{dk}$$

for $k \neq 0$, meaning dispersion is present. How this manifests itself by the spreading of the data and the resulting decay can be easily made explicit by directly computing the preceding Fourier integral.

Specifically, we have

Lemma 2.1. *The solution of the initial value problem for the Schrodinger equation*

$$i\partial_t\psi + \triangle\psi = 0,\ \psi(0, x) = f(x)$$

with $f \in \mathcal{S}(\mathbb{R})$ is given by

$$\psi(t, x) = \frac{1}{\sqrt{4\pi i t}}\int_{\mathbb{R}}e^{i\frac{(x-y)^2}{4t}}f(y)\,dy$$

In particular, we have

$$\left|\psi\right| \leq (4\pi|t|)^{-\frac{1}{2}} \cdot \left\|f\right\|_{L^1(\mathbb{R})}.$$

The profile of the initial data gets flattened and 'smeared out'.

Proof. Write

$$\begin{aligned}(2\pi)^{-1}\int_{\mathbb{R}} e^{i(x\cdot\xi - t\xi^2)}\widehat{f}(\xi)\,d\xi &= \lim_{\varepsilon\downarrow 0}(2\pi)^{-1}\int_{\mathbb{R}} e^{i(x\cdot\xi - t\xi^2)-\varepsilon\xi^2}\widehat{f}(\xi)\,d\xi\\ &= \lim_{\varepsilon\downarrow 0}(2\pi)^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i([x-y]\cdot\xi - t\xi^2)-\varepsilon\xi^2} f(y)\,dyd\xi\end{aligned}$$

In the last integral we are allowed to interchange the order of integration due to the absolute convergence, and we arrive at

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}} K_\varepsilon(x-y)f(y)\,dy,$$

where

$$K_\varepsilon(r) = (2\pi)^{-1}\int_{\mathbb{R}} e^{i(r\xi - t\xi^2)-\varepsilon\xi^2}\,d\xi.$$

This integral is standard to evaluate using basic complex analysis: write

$$\begin{aligned}i(r\xi - t\xi^2) - \varepsilon\xi^2 &= -i(t-i\varepsilon)\xi^2 + ir\xi\\ &= -i(a_\varepsilon\xi - b_\varepsilon)^2 + i\frac{r^2}{4(t-i\varepsilon)}.\end{aligned}$$

where

$$a_\varepsilon = (t-i\varepsilon)^{\frac{1}{2}},\, b_\varepsilon = \frac{r}{2(t-i\varepsilon)^{\frac{1}{2}}},$$

and we choose the square root $(t-i\varepsilon)^{\frac{1}{2}}$ such that $\lim_{\varepsilon\to 0}(t-i\varepsilon)^{\frac{1}{2}} = t^{\frac{1}{2}}$ for $t>0$. Then, assuming $t>0$ for now, we have that

$$\lim_{\varepsilon\downarrow 0} K_\varepsilon(r) = (2\pi)^{-1}t^{-\frac{1}{2}}e^{i\frac{r^2}{4t}}\cdot\int_{\mathbb{R}} e^{-iz^2}\,dz.$$

To evaluate the remaining integral on the right, we note that by Cauchy's theorem this remains unchanged if we rotate the real axis by $\frac{\pi}{4}$ clockwise, and so letting $\Gamma = \{\frac{\xi - i\xi}{\sqrt{2}},\, \xi\in\mathbb{R}\}$, we obtain

$$\int_{\mathbb{R}} e^{-iz^2}\,dz = \int_\Gamma e^{-iz^2}\,dz = \frac{1-i}{\sqrt{2}}\int_{\mathbb{R}} e^{-\xi^2}\,d\xi = \sqrt{\pi}\cdot\frac{1-i}{\sqrt{2}}.$$

This shows that

$$\lim_{\varepsilon\downarrow 0} K_\varepsilon(r) = \frac{1}{\sqrt{4\pi t}}e^{i\frac{r^2}{4t}}\cdot\frac{1-i}{\sqrt{2}},$$

This equals

$$\frac{1}{\sqrt{4\pi i t}}e^{i\frac{r^2}{4t}},$$

and in fact gives the precise meaning of the square root at the beginning, in case $t>0$. In case $t<0$, use that the transformation

$$\psi(t,x)\longrightarrow\overline{\psi(-t,x)}$$

carries solutions of the Schrodinger equation into solutions, and so the result in this case follows from the one for $t>0$.

□

Remark 2.2. Formally we can also arrive at this result by invoking the solution formula for the heat equation

$$\psi_t = \psi_{xx},$$

which is given by

$$\psi(t,x) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y)\,dy.$$

Considering (formally!) the function $\phi(t,x) := \psi(it,x)$. we get

$$i\partial_t\phi = -\psi_t(it,x) = -\psi_{xx}(it,x),$$

and further

$$\phi(t,x) = \frac{1}{\sqrt{4\pi it}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4it}} f(y)\,dy,$$

which is consistent with our formula above.

3. The linear KdV equation

The celebrated Korteweg-de-Vries equation arises in the modelisation of water waves in a narrow channel, and is given by

$$\psi_t + \psi_{xxx} - 6\psi\psi_x = 0,\ (t,x) \in \mathbb{R}^{1+1} \tag{3.1}$$

This is a nonlinear equation, but in the regime of very small amplitudes $|\psi| \ll 1$, it is natural to neglect the nonlinear interaction term $-6\psi\psi_x$ and simplify the equation to the following linear equation

$$\psi_t + \psi_{xxx} = 0. \tag{3.2}$$

It turns out that this is also a dispersive equation:

Lemma 3.1. *The model* (3.2) *on* $\mathbb{R}^{1+1}$ *is dispersive, and we have amplitude decay as follows: if*

$$\psi_t + \psi_{xxx} = 0,\ \psi(0,x) = f(x) \in \mathcal{S}(\mathbb{R}),$$

then there is a solution $\psi(t,x) \in C^\infty(\mathbb{R}^{1+1})$ *with* $\psi(t,\cdot) \in \mathcal{S}(\mathbb{R})$ *for all* $t \in \mathbb{R}$, *and such that*

$$\big|\psi(t,\cdot)\big| \le C|t|^{-\frac{1}{3}} \cdot \big\|f\big\|_{L^1(\mathbb{R}^3)},$$

where C *is a universal constant.*

Proof. We again use the Fourier representation for ψ, i. e. write

$$\psi(t,x) = (2\pi)^{-1}\int_{\mathbb{R}} \widehat{\psi}(t,\xi)\cdot e^{ix\xi}\,d\xi,$$

and infer the equation

$$\partial_t\widehat{\psi}(t,\xi) - i\xi^3\widehat{\psi}(t,\xi) = 0,\ \widehat{\psi}(0,\xi) = \widehat{f}(\xi).$$

Thus we can formally write

$$\psi(t,x) = (2\pi)^{-1}\int_{\mathbb{R}} e^{i(t\xi^3 + x\xi)}\widehat{f}(\xi)\,d\xi,$$

and it is straightforward to check that this function is in $C^\infty(\mathbb{R}^{1+1})$ and of Schwartz class for each fixed $t \in \mathbb{R}$. Also note that setting $k = \xi$, $\omega = -k^3$, the dispersion relation is again nonlinear, and the equation is indeed dispersive.

It remains to prove the dispersive decay estimate, which we do in analogy by interpreting the previous integral as a convolution of f with a kernel function K. We may assume $t > 0$, the other situation being handled similarly. Using a damping function $e^{-\varepsilon\xi^2}$ as in the previous section and passing to the limit $\varepsilon \to 0$, we infer that for $t \neq 0$

$$\psi(t,x) = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} K_{t,\varepsilon}(x-y)f(y)\,dy,$$

where

$$K_{t,\varepsilon}(z) := \int_R e^{i(t\xi^3 + z\xi) - \varepsilon\xi^2}\,d\xi.$$

It is straightforward to check that for $t \neq 0$

$$\lim_{\varepsilon \downarrow 0} K_{t,\varepsilon}(z) = \int_R e^{i(t\xi^3 + z\xi)}\, d\xi =: K_t(z)$$

and the limit is locally uniform with respect to z. Observe that

$$K_t(z) = t^{-\frac{1}{3}} \cdot \int_{\mathbb{R}} e^{i(\eta^3 + \frac{z}{t^{\frac{1}{3}}}\eta)}\, d\eta.$$

Then we make the

Claim: *the function* $r \longrightarrow \int_{\mathbb{R}} e^{i(\eta^3 + r\eta)}\, d\eta$ *is* C^∞ *and bounded.*

Once the claim is proven, the dispersive estimate of the lemma follows immediately.

To prove the claim, we shall use integration by parts. If $|r| < 1$, write

$$\int_{\mathbb{R}} e^{i(\eta^3 + r\eta)}\, d\eta = \int_{\mathbb{R}} \chi(\eta) e^{i(\eta^3 + r\eta)}\, d\eta + \int_{\mathbb{R}} (1 - \chi(\eta)) e^{i(\eta^3 + r\eta)}\, d\eta,$$

where $\chi \in C_0^\infty(\mathbb{R})$ is equals 1 on $[-2, 2]$. Then we have

$$\Big| \int_{\mathbb{R}} \chi(\eta) e^{i(\eta^3 + r\eta)}\, d\eta \Big| \leq C_1,$$

simply because $|e^{i(\eta^3 + r\eta)}| = 1$.
For the remaining integral, we write

$$e^{i(\eta^3 + r\eta)} = \frac{1}{i(3\eta^2 + r)} \partial_\eta \big(e^{i(\eta^3 + r\eta)} \big),$$

and observe that due to our assumption on r, the term $3\eta^2 + r \neq 0$ on the support of $1 - \chi(\eta)$. Then we can write

$$\int_{\mathbb{R}} (1 - \chi(\eta)) e^{i(\eta^3 + r\eta)}\, d\eta = -\int_{\mathbb{R}} \partial_\eta \big((1 - \chi(\eta)) \cdot \frac{1}{i(3\eta^2 + r)} \big) e^{i(\eta^3 + r\eta)}\, d\eta,$$

and since we have the easily verified bound

$$3\eta^2 + r \geq \eta^2$$

for $\eta \in \operatorname{supp}(1 - \chi(\eta))$, and furthermore (check!)

$$\big| \partial_\eta \big((1 - \chi(\eta)) \cdot \frac{1}{i(3\eta^2 + r)} \big) \big| \leq \frac{C_2}{3\eta^2 + r},$$

we see that

$$\big| \int_{\mathbb{R}} (1 - \chi(\eta)) e^{i(\eta^3 + r\eta)}\, d\eta \big| \leq C_2 \int_{\mathbb{R} \setminus [-1,1]} \eta^{-2}\, d\eta \leq C_3.$$

Obtaining a uniform bound for general $|r| \geq 1$ is handled analogously. Fix such a r. Then let $\chi_l(x) \in C_0^\infty(\mathbb{R})$, $l = 0, 1, \ldots, \lfloor \log |r| \rfloor + 1$ be non-negative functions such that

$$\sum_{0 \leq l \leq \log |r|} \chi_l(x) = 1,\ \chi_l(x) \neq 0 \to |x| \sim 2^l,\ |\chi'(x)| \leq C 2^{-l},\ l \geq 1,$$

where the first relation holds for $|x| \leq 2|r|$, and $\chi_0 = \chi$ is the function from above. The construction of such functions is relegated to the exercises. Then consider the integral

$$\int_{\mathbb{R}} \big(\sum_{0 \leq l \leq \lfloor \log |r| \rfloor + 1} \chi_l(3\eta^2 + r) \big) e^{i(\eta^3 + r\eta)}\, d\eta$$

Notice that the sets of η with $|3\eta^2 + r| \leq C$ is inside two intervals of length $\lesssim \frac{1}{\sqrt{|r|}}$. In fact, if $\eta_{1,2}$ satisfy the inequality and are both positive, then

$$\big|\eta_1 - \eta_2\big| = \frac{\big|3\eta_1^2 - 3\eta_2^2\big|}{3\eta_1 + 3\eta_2} \leq D|r|^{-\frac{1}{2}}$$

This shows that

$$\big|\int_{\mathbb{R}} \chi_0(3\eta^2 + r)e^{i(\eta^3 + r\eta)}\, d\eta\big| \leq 2D \cdot |r|^{-\frac{1}{2}}$$

For the remaining sum $\sum_{1\leq l\leq \lfloor \log |r| \rfloor + 1}$, we perform integration by parts. This leads to the integral

$$-\int_{\mathbb{R}} \partial_\eta \Big(\sum_{1\leq l\leq \lfloor \log |r| \rfloor + 1} \frac{\chi_l(3\eta^2 + r)}{i(3\eta^2 + r)} \Big) e^{i(\eta^3 + r\eta)}\, d\eta$$

Now the set of η with $\chi_l(3\eta^2 + r) \neq 0$ is contained in two intervals of length $\leq E \cdot \frac{2^l}{\sqrt{|r|}}$. Then we can estimate for $l \geq 1$

$$\big|\partial_\eta \big(\frac{\chi_l(3\eta^2 + r)}{i(3\eta^2 + r)} \big)\big| \leq E \cdot \frac{|\eta|}{(3\eta^2 + r)^2} \leq F \cdot |r|^{\frac{1}{2}} \cdot 2^{-2l}.$$

It then follows that

$$\begin{aligned} &\big| - \int_{\mathbb{R}} \partial_\eta \big(\frac{\chi_l(3\eta^2 + r)}{i(3\eta^2 + r)} \big) \cdot e^{i(\eta^3 + r\eta)}\, d\eta \big| \\ &\leq 2E \cdot \frac{2^l}{\sqrt{|r|}} \cdot F \cdot |r|^{\frac{1}{2}} \cdot 2^{-2l} \leq G \cdot 2^{-l}. \end{aligned}$$

Summing over $1 \leq l \leq \lfloor \log |r| \rfloor + 1$ results in

$$\big| - \int_{\mathbb{R}} \partial_\eta \Big(\sum_{1\leq l\leq \lfloor \log |r| \rfloor + 1} \frac{\chi_l(3\eta^2 + r)}{i(3\eta^2 + r)} \Big) e^{i(\eta^3 + r\eta)}\, d\eta \big| \leq H \cdot \sum_{1\leq l\leq \lfloor \log |r| \rfloor + 1} 2^{-l} \leq 2H.$$

We have now reduced the proof of the boundedness part of the claim to estimating

$$\int_{\mathbb{R}} \big(1 - \sum_{0\leq l\leq \lfloor \log |r| \rfloor + 1} \chi_l(3\eta^2 + r)\big) e^{i(\eta^3 + r\eta)}\, d\eta.$$

Here $3\eta^2 + r \geq 2|r|$ on the support of the integrand, and one again easily obtains the desired bound via integration by parts; we leave the details, as well as the proof of C^∞-smoothness, to the exercises.

□