FOURIER MULTIPLIERS; MIKHLIN MULTIPLIER THEOREM

We follow Muscalu-Schlag Vol. I.

1. Introduction

Up until now we have studies operators which were given as convolution integrals $Tf(x) = \int_{\mathbb{R}^n} K(x-y)f(y)\,dy$, and deduced their L^p-boundedness in terms of suitable conditions on K. However, many operators, such as the Riesz operators $R_j f = \partial_{x_j}\sqrt{-\triangle}^{-1} f$, can be conventiently expressed in terms of the Fourier transform, i. e. we often encounter operators of the form

$$Tf(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} m(\xi)\widehat{f}(\xi)\,d\xi. \tag{1.1}$$

For example, the Riesz multipliers are given by the Fourier multiplier $m(\xi) = \frac{\xi_j}{|\xi|}$, and the Hilbert transform (on $\mathbb{R} = \mathbb{R}^1$) is given by multiplier $m(\xi) = c\text{sign}\xi$. It is then very natural to develop a theory of L^p-boundedness of such operators in terms of suitable conditions on $m(\xi)$. Fortunately, we shall be able to develop such a theory quite easily based on our earlier studies of Calderon-Zygmund operators.

2. Localization to dyadic scales

An extremely useful and versatile tool in studying operators of the form (1.1) is *localization to dyadic scales*. Here there is nothing sacrosanct about *dyadic* scales $\xi \sim 2^k$, $k \in \mathbb{Z}$, it being possible to replace this by scales of the form $\xi \sim a^k$ for any $a > 1$. What matters is that the scales grow or shrink exponentially with k, which enables summation over these scales in many situations. However, dyadic is the generally used standard, so we adhere to it.
The very first step of so-called *Littlewood-Paley calculus*, which is the study of the properties of functions in terms of their dyadic parts, is the introduction of a suitable *partition of unity* subordinate to dyadic scales, as in the following

Lemma 2.1. *There exists a radial function $\psi \in C_0^\infty(\mathbb{R}^n)$ supported on $\mathbb{R}^n\backslash\{0\}$ and with the property that*

$$\sum_{j\in\mathbb{Z}} \psi(\frac{x}{2^j}) = 1\,\forall x \in \mathbb{R}^n\backslash\{0\}.$$

Also, we may ensure that for any $x \in \mathbb{R}^n\backslash\{0\}$ there are at most two non-zero terms in the sum on the left.

Proof. Pick a radial function $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ with $\chi(x) = 1$, $|x| \leq 1$, and $\chi(x) = 0$, $|x| \geq 2$. Further, let

$$\psi(x) := \chi(x) - \chi(2x)$$

Note that $\psi(x) = 0$ unless $\frac{1}{2} < |x| < 2$. Thus, if $\psi(\frac{x}{2^j}) \neq 0$, then depending on whether $1 \leq \frac{x}{2^j} < 2$ or $\frac{1}{2} < \frac{x}{2^j} < 1$, we we can have $\psi(\frac{x}{2^{j+1}}) \neq 0$ or $\psi(\frac{x}{2^{j-1}}) \neq 0$ but $\psi(\frac{x}{2^k}) = 0$ for all other values of k. Thus we have the final assertion of the lemma for this choice of ψ. Moreover, we have

$$\sum_{j=-N}^{j=N} \psi(\frac{x}{2^j}) = \chi(\frac{x}{2^N}) - \chi(2^{N+1}x)$$

Given $x \neq 0$, pick N large enough such that $\frac{|x|}{2^N} \leq 1$ and $2^{N+1}|x| > 2$. Thus we get

$$\sum_{j\in\mathbb{Z}} \psi(\frac{x}{2^j}) = 1\,\forall x \in \mathbb{R}^n\backslash\{0\}.$$

□

The cutoffs $\psi(\frac{x}{2^j})$ of the preceding lemma allow us to introduce the so-called Littlewood-Paley localizers P_j: for any $f \in \mathcal{S}(\mathbb{R}^n)$, say, we put

$$\widehat{P_j f}(\xi) := \psi(\frac{\xi}{2^j})\widehat{f}(\xi).$$

Then the preceding lemma easily implies that

$$\sum_{j\in\mathbb{Z}} P_j f = f.$$

In many situations it is extremely useful to replace a function f by its dyadically localized pieces $P_j f$, $j \in \mathbb{Z}$, as this gives much better control, and one eventually sums over the different dyadic scales. Such a procedure will be quite successful in the proof of the following theorem.

3. L^p-bounds for Fourier multipliers

We now prove a quite general theorem that implies good L^p-bounds for a wide variety of Fourier mutipliers, which in particular encompasses all Riesz multipliers. This is the *Mikhlin multiplier theorem*

Theorem 3.1. *(Mikhlin) Let $m : \mathbb{R}^n\backslash\{0\} \to \mathbf{C}$ satisfy the bounds*

$$\left|\partial_\xi^\gamma m(\xi)\right| \leq B|\xi|^{-|\gamma|},\ |\gamma| \leq n+2,\ \xi \neq 0.$$

for any multi-index γ of length $\leq n+2$. Then the corresponding Fourier multiplier satisfies the bounds

$$\left\|\mathcal{F}^{-1}(m\widehat{f})\right\|_{L^p(\mathbb{R}^n)} \leq C(n,p)\cdot B\left\|f\right\|_{L^p(\mathbb{R}^n)},\ 1 < p < \infty,$$

Proof. This is an application of the dyadic localization technique. Thus we instead consider the localized operators

$$T_j f := TP_j f = \mathcal{F}^{-1}(m\psi(\frac{\xi}{2^j})\widehat{f}),$$

where we have replaced the kernel $m(\xi)$ by the localized $m(\xi)\psi(\frac{\xi}{2^j})$. Then our strategy will be to show that each of the T_j is in fact a strong Calderon Zygmund operator with kernel bounds which decay suitably in relation to j and can be summed up to show that the original operator $T = \mathcal{F}^{-1}(m\widehat{f})$ satisfies all the required properties in C.-Z. theory to give L^p-boundedness.

Thus write

$$\mathcal{F}^{-1}(m\psi(\frac{\xi}{2^j})\widehat{f}) = \int_{\mathbb{R}^n} K_j(x-y)f(y)\,dy,$$

where $K_j(x) = \mathcal{F}^{-1}(m(\xi)\psi(\frac{\xi}{2^j}))$. To bound K_j, note that for any $k \in [0, n+2]$, we have

$$|x^k|\left|K_j(x)\right| \leq C\sum_{|\gamma|=k}\left|x^\gamma K_j(x)\right| = C_1\sum_{|\gamma|=k}\left|\mathcal{F}^{-1}\left(\partial_\xi^\gamma m_j(\xi)\right)\right|$$

Thus we get

$$\left\||x^k|\left|K_j(x)\right|\right\|_{L^\infty(\mathbb{R}^n)} \leq C_1\sum_{|\gamma|=k}\left\|\partial_\xi^\gamma m_j\right\|_{L^1(\mathbb{R}^n)} \leq C_2\cdot B2^{(n-k)j}$$

or in other words

$$|K_j(x)| \leq C_2\cdot B2^{(n-k)j}|x|^{-k},\ k \in [0, n+2]$$

Similarly, since differentiation with respect to x translates into multiplication with ξ up to a constant, we get

$$|\nabla K_j(x)| \leq C_2\cdot B2^{(n+1-k)j}|x|^{-k},\ k \in [0, n+2].$$

The strategy now is to infer similar bounds for the sum $K(x) = \sum_j K_j(x)$ by using the preceding bounds with suitably chosen $k \in [0, n+2]$. Thus we get for $x \neq 0$

$$\left|K(x)\right| \leq \sum_{2^j<|x|^{-1}} C_2\cdot B2^{nj} + \sum_{2^j>|x|^{-1}} C_2\cdot B2^{-j}|x|^{-n-1} \leq C_3\cdot B|x|^{-n}$$

Also

$$\big|\nabla K(x)\big| \leq \sum_{2^j<|x|^{-1}} C_2\cdot B2^{(n+1)j} + \sum_{2^j>|x|^{-1}} C_2\cdot B2^{-j}|x|^{-n-2} \leq C_3\cdot B|x|^{-n-1}$$

Note that in the last sum we used $k = n+2$.

The proof of the theorem is now straightforward. We know that the condition on ∇K characteristic for a strong C.-Z. operator implies the Hormander condition, and we also have the required pointwise bound on K for a C.-Z. operator. We also know from the proof of the weak L^1-bound for C.-Z. operators that we only need the L^2-boundedness in addition to the preceding two conditions to give all L^p-bounds for $1 < p < \infty$. The L^2-bound, however, is obvious, since by Plancherel's theorem we have

$$\big\|T\big\|_{L^2\to L^2} \leq \big\|m\big\|_{L^\infty} \leq B.$$

□

Corollary 3.2. *We have the bounds*

$$\big\|R_j f\big\|_{L^p(\mathbb{R}^n)} \leq C_p\big\|f\big\|_{L^p(\mathbb{R}^n)},\ j=1,2,\ldots,n,\ 1<p<\infty$$

In particular, if

$$\triangle f = h$$

with $h \in \mathcal{S}(\mathbb{R}^n)$, say, and $\lim_{|x|\to\infty} f(x) = 0$, then we have

$$\big\|\nabla^2 f\big\|_{L^p(\mathbb{R}^n)} := \sum_{1\leq i,j\leq n} \big\|\partial^2_{x_i x_j} f\big\|_{L^p(\mathbb{R}^n)} \leq C_p\big\|h\big\|_{L^p(\mathbb{R}^n)}, 1<p<\infty.$$

Proof. The first part follows since $\widehat{R_j f} = \frac{\xi_j}{|\xi|}\widehat{f}(\xi)$, and the symbol $m(\xi) := \frac{\xi_j}{|\xi|}$ satisfies the properties of the Mikhlin multiplier theorem (in fact for any $|\gamma| \geq 0$).
The second part follows since

$$\partial^2_{x_i x_j} f = -R_i R_j h$$

□

4. More general Fourier multipliers

The issue of L^p-boundedness of Fourier multipliers of the form

$$Tf(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} m(\xi)\widehat{f}(\xi)\,d\xi$$

hinges delicately on the differentiability properties of the multiplier m. The preceding proof revealed that we may allow a singularity of m at a point (here $\xi = 0$) provided we carefully control the growth of the derivatives of m as we approach the singularity. Note that this also encompasses the Hilbert transform with multiplier $c\frac{\xi}{|\xi|}$. But what happens with 'more singular' symbols, such as the ball multiplier

$$m(\xi) := \chi_B(\xi)$$

where $B = B_1(0) \subset \mathbb{R}^n$, $n \geq 2$, say, and χ_B denotes the characteristic function of this set? Here $m(\xi)$ is singular along $\partial B = S^{n-1}$, and our preceding method of proof breaks down. In fact, here we have the *complete failure* of L^p-boundedness:

Theorem 4.1. *(C. Fefferman 1971) For any $p \neq 2$, the Fourier multiplier χ_B on functions in $\mathbb{R}^n$, $n \geq 2$, is not bounded. Thus for any $M > 0$ and $p \neq 2$, there is $f \in \mathcal{S}(\mathbb{R}^n)$, $f \neq 0$, and such that*

$$\big\|\mathcal{F}^{-1}\big(\chi_B(\xi)\widehat{f}(\xi)\big)\big\|_{L^p(\mathbb{R}^n)} \geq M\big\|f\big\|_{L^p(\mathbb{R}^n)}.$$

A natural variation here is to consider the 'slightly less singular' multiplier

$$m_\delta(\xi) := (1 - |\xi|^2)_+^\delta = (1 - |\xi|^2)_+^\delta \chi_B(\xi),\ \delta > 0$$

It turns out that then one gets a certain range of p around $p = 2$ for which one does have L^p-boundedness, and in fact the optimal result is known in $n = 2$ dimensions (due to Carleson-Sjolin), but the optimal result for $n \geq 3$ is open as of this moment in time (Bochner-Riesz Conjecture).