THE ENDPOINT ESTIMATE FOR CALDERON-ZYGMUND OPERATORS

We follow the treatment in Muscalu-Schlag, Vol 1.

1. Calderon-Zygmund decomposition of a function

Here we introduce a fundamental technical tool in Harmonic Analysis on $\mathbb{R}^n$, which allows us to decompose general functions $f \in L^1(\mathbb{R}^n)$ into a 'good part' in the sense that it is bounded and a 'bad part', which is, however, restricted to a relatively small set.

Theorem 1.1. *(Calderon-Zygmund) Let $f \in L^1(\mathbb{R}^n)$, and fix $\lambda > 0$. Then we can write*

$$f = g + b$$

with $|g| \leq \lambda$ a.e. and $b = \sum_Q \chi_Q f$ with the sum running over a disjoint collection of cubes and such that

$$\lambda < |Q|^{-1} \int_Q |f(x)| dx < 2^n \lambda$$

Moreover, we can bound the support of the 'bad part' b by

$$\big| \cup Q \big| \leq \lambda^{-1} \big\| f \big\|_{L^1(\mathbb{R}^n)}.$$

Proof. First, we partition $\mathbb{R}^n$ for every $k \in \mathbb{Z}$ into a (almost) disjoint collection of (closed) dyadic cubes $Q \in \mathcal{D}_l$ as follows: these have vertices in the points

$$(2^l k_1, 2^l k_2, \ldots, 2^l k_n),\ k_j \in \mathbb{Z}$$

and edges parallel to the coordinates and of length 2^l.
We now begin an inductive process defining the cubes Q in the Calderon-Zygmund decomposition. First, pick $l_* \in \mathbb{Z}$ large enough such that

$$2^{-nl_*} \int_{\mathbb{R}^n} |f| \, dx \leq \lambda.$$

Then we have

$$|Q|^{-1} \int_Q |f| \, dx \leq \lambda$$

for every cube $Q \in \mathcal{D}_{l*}$. Now for each cube $Q \in \mathcal{D}_{l*}$ look at its 'children', obtained by passing from l_* to $l_* - 1$ and taking those cubes $Q' \in \mathcal{D}_{l*-1}$ with $Q' \subset Q$. Then if we have

$$|Q'|^{-1} \int_{Q'} |f| \, dx > \lambda$$

for such a cube, we stop the process and add the cube Q' to the collection of 'bad cubes'. Otherwise, replace Q by Q' and restart the process, i. e. pass to $l_* - 2$, consider the dyadic children of Q' and proceed as before. Taking the collection $\mathcal{B}$ of all bad cubes where the process stopped, we then set

$$b := \sum_{Q \in \mathcal{B}} \chi_Q f,$$

where the sum converges in the L^1-sense, and further

$$g := f - \sum_{Q \in \mathcal{B}} \chi_Q f$$

Now if $x \in \mathbb{R}^n \setminus \cup_{\mathcal{B}} Q$, then by definition there is a sequence of shrinking cubes $Q_i \to x$ and for each of these we have

$$|Q_i|^{-1} \int_{Q_i} |f| \, dx \leq \lambda.$$

By the Lebesgue differentiation theorem, we have

$$\lim_{i\to\infty} |Q_i|^{-1} \int_{Q_i} |f|\,dx = |f(x)|$$

for almost every x and any shrinking sequence of cubes Q_i converging to x. It follows that

$$|g(x)| \le \lambda$$

almost everywhere.
Also, if Q is a 'bad cube', then its 'parent' $\widetilde{Q}$ which is the dyadic cube of twice its edge length and containing it is by definition such that

$$|\widetilde{Q}|^{-1} \int_{\widetilde{Q}} |f|\,dx \le \lambda,$$

and so we have

$$|Q|^{-1} \int_{Q} |f|\,dx \le 2^n \cdot |\widetilde{Q}|^{-1} \int_{\widetilde{Q}} |f|\,dx \le 2^n \lambda.$$

Finally, we have

$$\big|\cup_{\mathcal{B}} Q\big| \le \sum_{\mathcal{B}} \big|Q\big| \le \lambda^{-1} \sum_{Q\in\mathcal{B}} \int_{Q} |f|(x)\,dx \le \lambda^{-1} \big\|f\big\|_{L^1(\mathbb{R}^n)},$$

where we have used the (almost) disjointness of the cubes Q.

□

2. The weak endpoint bound for Calderon-Zygmund operators

Using the preceding result, we shall now prove

Theorem 2.1. *Let T be a Calderon-Zygmund operator, given by*

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y) f(y)\,dy$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. Then we have the weak L^1-bound

$$\big|\{x \in \mathbb{R}^n \mid |Tf(x)| > \lambda\}\big| \le CB\lambda^{-1} \big\|f\big\|_{L^1(\mathbb{R}^n)} \ \forall \lambda > 0$$

for a suitable universal constant $C = C(n)$ and B as in the definition of Calderon-Zygmund kernel.

Proof. Replacing T by $B^{-1}T$, it suffices to assume $B = 1$. In fact, assume we have the bound in the case $B = 1$. Then we have

$$\{|Tf(x)| > \lambda\} = \{\big|B^{-1}Tf(x)\big| > B^{-1}\lambda\},$$

where $B^{-1}T$ is a C.Z.-kernel with $B = 1$, and so we get

$$\begin{aligned}\big|\{\big|Tf(x)\big| > \lambda\}\big| = \big|\{\big|B^{-1}Tf(x)\big| > B^{-1}\lambda\}\big| &\le C(n) \cdot (B^{-1}\lambda)^{-1} \cdot \big\|f\big\|_{L^1(\mathbb{R}^n)} \\ &= C(n)B\lambda^{-1} \cdot \big\|f\big\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

Fix $\lambda > 0$, $f \in \mathcal{S}(\mathbb{R}^n)$, and implement the corresponding Calderon-Zygmund decomposition of f, $f = g + b$ associated to λ. In fact, we modify this a bit, so that the 'bad part' also has an important vanishing property: write

$$f = f_1 + f_2,$$

where we set

$$f_1 = g + \sum_{\mathcal{B}} \chi_Q(x) |Q|^{-1} \int_Q f(y)\,dy,\ f_2 = b - \sum_{\mathcal{B}} \chi_Q(x) |Q|^{-1} \int_Q f(y)\,dy.$$

Then we observe the following

- The function f_1 is bounded. In fact, we have

$$\big|f_1\big| \le \max\{\big|g\big|, \sup_{Q\in\mathcal{B}} |Q|^{-1} \int_Q |f(x)|\,dx\} \le 2^n \lambda.$$

- Both functions are bounded in $L^1(\mathbb{R}^n)$. In fact we have

$$\|f_1\|_{L^1(\mathbb{R}^n)} \leq \|g\|_{L^1} + \sum_{Q\in\mathcal{B}} \int_Q |f(x)|\,dx = \|f\|_{L^1},$$

while

$$\|f_2\|_{L^1} \leq \|b\|_{L^1} + \sum_{Q\in\mathcal{B}} \int_Q |f(x)|\,dx \leq 2\|f\|_{L^1}.$$

- The function f_2 has vanishing average over each $Q \in \mathcal{B}$. In fact, we have for $Q \in \mathcal{B}$

$$\int_Q f_2(x)\,dx = \int_Q (f(x) - |Q|^{-1}\int_Q f(y)\,dy)\,dx = 0.$$

We shall now use two different arguments to handle $f_{1,2}$. To begin with, observe that since

$$Tf = Tf_1 + Tf_2,$$

we have

$$\{x \in \mathbb{R}^n \,|\, |Tf(x)| > \lambda\} \subset \{x \in \mathbb{R}^n \,|\, |Tf_1(x)| > \frac{\lambda}{2}\} \cup \{x \in \mathbb{R}^n \,|\, |Tf_2(x)| > \frac{\lambda}{2}\},$$

and so it suffices to show that

$$\big|\{x \in \mathbb{R}^n \,|\, |Tf_j(x)| > \lambda\}\big| \leq C\lambda^{-1}\|f\|_{L^1},\ j = 1, 2.$$

The argument for f_1. This is simpler, since we can invoke the L^2-boundedness of T which we have already proved. Thus we have

$$\big|\{x \in \mathbb{R}^n \,|\, |Tf_1(x)| > \lambda\}\big| \leq \lambda^{-2}\int_{\mathbb{R}^n}[Tf_1(x)]^2\,dx \leq C\lambda^{-2}\|f_1\|_{L^2(\mathbb{R}^n)}^2$$

From the preceding, we know that $\|f_1\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$, and using the L^∞-bound $\|f_1\| \leq 2^n\lambda$, we can pass to

$$\|f_1\|_{L^2}^2 = \int_{\mathbb{R}^n} |f_1|\cdot|f_1|\,dx \leq 2^n\lambda\int_{\mathbb{R}^n}|f_1|(x)\,dx \leq 2^n\lambda\|f\|_{L^1},$$

and so we get

$$\big|\{x \in \mathbb{R}^n \,|\, |Tf_1(x)| > \lambda\}\big| \leq C_1\lambda^{-1}\|f\|_{L^1},$$

as desired.

The argument for f_2. This is a bit more complicated. For each bad cube $Q \in \mathcal{B}$, pick a sufficiently large dilate Q_*, such that

$$|x - y| \geq 2|y - y_1|\,\forall y, y_1 \in Q$$

and where $x \in (Q_*)^c$ is arbitrary. We can immediately reduce to bounding the set

$$\big|\{x \in \mathbb{R}^n \setminus \cup_{\mathcal{B}} Q_* \,|\, |Tf_2(x)| > \lambda\}\big|$$

on account of $\big|\cup_{\mathcal{B}} Q_*\big| \leq \sum_{\mathcal{B}}|Q_*| \leq C\lambda^{-1}\|f\|_{L^1}$. For $x \in \mathbb{R}^n \setminus \cup_{\mathcal{B}} Q_*$, write

$$\begin{aligned} Tf_2(x) = \int_{\mathbb{R}^n} K(x-y)f_2(y)\,dy &= \sum_{\mathcal{B}}\int_{\mathbb{R}^n} K(x-y)f_Q(y)\,dy \\ &= \sum_{\mathcal{B}}\int_Q K(x-y)f_Q(y)\,dy \end{aligned}$$

where we have set $f_Q(y) := \chi_Q(y)[f(y) - |Q|^{-1}\int_Q f(x)\,dx]$.
But then using the vanishing property of f_Q we have

$$\int_Q K(x-y)f_Q(y)\,dy = \int_Q [K(x-y) - K(x-y_Q)]f_Q(y)\,dy$$

where y_Q is the centre of Q. We conclude that

$$\big|\{x \in \mathbb{R}^n \setminus \cup_{\mathcal{B}} Q_* \,|\, |Tf_2(x)| > \lambda\}\big| \leq \lambda^{-1} \int_{\mathbb{R}^n \setminus \cup_{\mathcal{B}} Q_*} |Tf_2(x)|\, dx$$

$$\leq \lambda^{-1} \int_{\mathbb{R}^n \setminus \cup_{\mathcal{B}} Q_*} \sum_{\mathcal{B}} \int_Q |K(x-y) - K(x-y_Q)||f_Q(y)|\, dydx$$

Switching the order of integration, we have

$$\int_{\mathbb{R}^n \setminus \cup_{\mathcal{B}} Q_*} |K(x-y) - K(x-y_Q)|\, dx \leq C$$

by the properties of the kernel K, and so

$$\lambda^{-1} \int_{\mathbb{R}^n \setminus \cup_{\mathcal{B}} Q_*} \sum_{\mathcal{B}} \int_Q |K(x-y) - K(x-y_Q)||f_Q(y)|\, dydx$$

$$\leq C\lambda^{-1} \sum_{\mathcal{B}} \int_Q |f_Q(y)|\, dy \leq C\lambda^{-1} \|f\|_{L^1},$$

as desired.

□

3. L^p-boundedness for Calderon-Zygmund operators

By Marcinkiewicz interpolation the preceding sections imply the boundedness of C.-Z. operators on all spaces $L^p(\mathbb{R}^n)$, $1 < p \leq 2$. By duality, we extend that to all $1 < p < \infty$. In fact, if $2 \leq p < \infty$, observe that

$$\|T(f)\|_{L^p(\mathbb{R}^n)} = \sup_{\|g\|_{L^{p'}} \leq 1} \Big| \int_{\mathbb{R}^n} T(f)(x)\overline{g(x)}\, dx \Big|$$

$$= \sup_{\|g\|_{L^{p'}} \leq 1} \Big| \int_{\mathbb{R}^n} f(x)\overline{T^* g(x)}\, dx \Big|$$

where we define the dual C.-Z. operator T^* by

$$T^*(g)(x) := \int_{\mathbb{R}^n} \overline{K}(y-x) g(y)\, dy.$$

It is immediate to verify that T^* is C.-Z. provided T is,and so we have

$$\|T^*(g)\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|g\|_{L^{p'}(\mathbb{R}^n)}.$$

We conclude via Holder's inequality that

$$\sup_{\|g\|_{L^{p'}} \leq 1} \Big| \int_{\mathbb{R}^n} f(x)\overline{T^* g(x)}\, dx \Big| \leq \|f\|_{L^p(\mathbb{R}^n)} \|T^* g\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},$$

as desired.