SINGULAR INTEGRAL OPERATORS IN HIGHER DIMENSIONS; CALDERON-ZYGMUND OPERATORS

We follow the treatment in Muscalu-Schlag, Vol 1.

1. Calderon Zygmund kernels

The Hilbert transform in one dimension is given by the Fourier multiplier $m(\xi) = -i\pi\text{sign}(\xi) = -i\pi\frac{\xi}{|\xi|}$, meaning that

$$H(f) = \int_{\infty}^{\infty} e^{2\pi i x\cdot\xi} m(\xi)\widehat{f}(\xi)\, d\xi.$$

There are very natural analogues of the Hilbert transform, so-called Riesz multipliers R_j, on higher dimensional $\mathbb{R}^n$, which are given by Fourier multipliers $\frac{\xi_j}{|\xi|}$ (where now $|\xi|$ denotes the Euclidean length), and so we have

$$R_j(f) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\frac{\xi_j}{|\xi|}\widehat{f}(\xi)\, d\xi,$$

where now f is a function on $\mathbb{R}^n$. Such operators occur naturally in lots of areas of PDE, and their boundedness properties in analogy to those of the Hilbert transform are of great importance.
Here we shall develop a general theory of singular integral operators given in terms of convolution with certain kernels, in analogy to the kernel $\frac{1}{x-y}$ for the Hilbert transform, and such that these operators do have good boundedness properties in L^p-spaces. Our theory shall also encompass Riesz multipliers.

Definition 1.1. *We call a function $K : \mathbb{R}^n\backslash\{0\} \to \mathbf{C}$ a Calderon-Zygmund kernel, provided the following three properties obtain:*

(1) *We have the bound $|K(x)| \leq B|x|^{-n}$ for some $B \in \mathbb{R}$.*
(2) *We have the bound*

$$\int_{|x|>2|y|} |K(x) - K(x-y)|\, dx \leq B$$

for all $y \neq 0$ and a fixed B.
(3) *We have the following cancellation property:*

$$\int_{r<|x|<s} K(x)\, dx = 0\,\forall r, s > 0$$

The somewhat complicated second condition in this definition, called *Hormander condition*, is in fact ensured by a more elementary condition, called *strong Calderon-Zygmund condition*: assume K differentiable on $\mathbb{R}^n\backslash\{0\}$ and

$$|\nabla K(x)| \leq B|x|^{-n-1}$$

In fact, we have

Lemma 1.2. *The preceding condition implies the Hormander condition.*

Proof. Using the fundamental theorem of calculus, write

$$K(x) - K(x-y) = -\int_0^1 \nabla_x K(x-ty)\cdot y\, dt$$

and so for $|y| < \frac{|x|}{2}$ we have

$$\big|K(x) - K(x-y)\big| \leq B|y|(\frac{|x|}{2})^{-n-1} = 2^{n+1}B|y||x|^{-n-1}.$$

Then we get

$$\int_{|x|>2|y|} |K(x) - K(x-y)|\,dx \le 2^{n+1}B|y| \int_{|x|>2|y|} |x|^{-n-1}\,dx$$
$$\le B_1|y| \cdot |y|^{-1} = B_1,$$

which gives Hormander's condition with a constant B_1. $\square$

A Calderon-Zygmund operator is an operator given by the formal expression

$$Tf(x) := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \chi_{|x-y|>\varepsilon} K(x-y) f(y)\,dy,$$

where K is a Calderon-Zygmund operator. That this is indeed a well-defined operator, at least when $f \in \mathcal{S}(\mathbb{R}^n)$, follows from

Lemma 1.3. *The expression $Tf(x)$ is well-defined as long as $f \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. Write

$$\begin{aligned}
&\int_{\mathbb{R}^n} \chi_{|x-y|>\varepsilon} K(x-y) f(y)\,dy \\
&= \int_{\mathbb{R}^n} \chi_{|y|>\varepsilon} K(y) f(x-y)\,dy \\
&= \int_{\mathbb{R}^n} \chi_{1>|y|>\varepsilon} K(y)[f(x-y) - f(x)]\,dy + \int_{\mathbb{R}^n} \chi_{|y|>1} K(y) f(x-y)\,dy,
\end{aligned}$$

where we have taken advantage of the cancellation property (1). But then

$$\big|K(y)[f(x-y) - f(x)]\big| \le B\big\|\nabla f\big\|_{L^\infty} |y|^{-n+1},$$

which is an absolutely integrable function near $y = 0$. It follows that the limit

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \chi_{1>|y|>\varepsilon} K(y)[f(x-y) - f(x)]\,dy$$

exists. $\square$

In the sequel, we shall develop a theory of L^p-boundedness of Calderon-Zygmund operators T as above. In fact, this shall be naturally divided into two parts, an easier L^2-boundedness part, and a harder 'almost L^1' type bound. It is for for the latter part that we shall have to introduce one of the main tools of the theory, the so-called Calderon-Zygmund decomposition of a function. Once these two 'endpoint estimates' are in place, the remaining L^p-estimates shall follow via Marcinkiewicz interpolation.

2. L^2-boundedness of Calderon-Zygmund operators

Here we prove the following

Proposition 2.1. *Let T be a Calderon-Zygmund operator given in terms of a Calderon-Zygmund kernel. There exists a constant $C = C(n)$ such that we have*

$$\big\|Tf\big\|_{L^2} \le CB\big\|f\big\|_{L^2}$$

In particular, T extends as a continuous operator to $L^2(\mathbb{R}^n)$.

Proof. For $r, s > 0$, introduce the 'truncated operator'

$$T_{r,s}(f)(x) := \int_{\mathbb{R}^n} K(y) \chi_{r<|y|<s} f(x-y)\,dy.$$

Throughout we shall assume $f \in \mathcal{S}(\mathbb{R}^n)$ to make everything well-defined. Recall that

$$\widehat{T_{r,s}(f)}(\xi) = \mathcal{F}\big(K(y)\chi_{r<|y|<s}\big)(\xi)\widehat{f}(\xi)$$

Due to Plancherel's theorem, it suffices to establish an L^∞-bound for the first factor on the right which is uniform in $r, s > 0$. In fact, we have

$$\begin{aligned}\big\|T_{r,s}(f)\big\|_{L^2(\mathbb{R}^n)} = \big\|\widehat{T_{r,s}(f)}\big\|_{L^2(\mathbb{R}^n)} &\leq \big\|\mathcal{F}\big(K(y)\chi_{r<|y|<s}\big)\big\|_{L^\infty(\mathbb{R}^n)}\big\|\widehat{f}(\xi)\big\|_{L^2(\mathbb{R}^n)} \\ &= \big\|\mathcal{F}\big(K(y)\chi_{r<|y|<s}\big)\big\|_{L^\infty(\mathbb{R}^n)}\big\|f\big\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

It follows that it suffices to show

$$\big\|\mathcal{F}\big(K(y)\chi_{r<|y|<s}\big)\big\|_{L^\infty(\mathbb{R}^n)} \leq CB \,\forall r, s > 0.$$

Observe that we have

$$\begin{aligned}\mathcal{F}\big(K(y)\chi_{r<|y|<s}\big) &= \int_{r<|y|<s} K(y)e^{-2\pi i y\cdot\xi}\,dy \\ &= \int_{r<|y|<\min\{s,|\xi|^{-1}\}} K(y)e^{-2\pi i y\cdot\xi}\,dy \\ &+ \int_{\max\{r,|\xi|^{-1}\}<|y|<s} K(y)e^{-2\pi i y\cdot\xi}\,dy \\ &:= A + B.\end{aligned}$$

We estimate each of these terms.

(i): the estimate for (A). Using the cancellation property (1), we get

$$\begin{aligned}|A| &= \Big|\int_{r<|y|<\min\{s,|\xi|^{-1}\}} K(y)[e^{-2\pi i y\cdot\xi} - 1]\,dy\Big| \\ &\leq 2\pi\int_{r<|y|<\min\{s,|\xi|^{-1}\}} |y||\xi||K(y)\,dy \\ &\leq 2\pi B|\xi|\int_{r<|y|<\min\{s,|\xi|^{-1}\}} |y|^{-n+1}\,dy \\ &\leq B_1,\end{aligned}$$

uniformly in $r, s > 0$.

(i): the estimate for (B). Here we exploit the cancellation condition (2) for the kernel K. Write

$$\int_{\max\{r,|\xi|^{-1}\}<|y|<s} K(y)e^{-2\pi i y\cdot\xi}\,dy = -\int_{\max\{r,|\xi|^{-1}\}<|y|<s} K(y)e^{-2\pi i(y+\frac{\xi}{2|\xi|^2})\cdot\xi}\,dy,$$

whence

$$\begin{aligned}2B &= \int_{\max\{r,|\xi|^{-1}\}<|y|<s}[K(y) - K(y - \frac{\xi}{2|\xi|^2})]e^{-2\pi i y\cdot\xi}\,dy \\ &- \int_{\substack{\max\{r,|\xi|^{-1}\}<|y-\frac{\xi}{2|\xi|^2}|<s \\ \backslash\{\max\{r,|\xi|^{-1}\}<|y|<s\}}} K(y - \frac{\xi}{2|\xi|^2})e^{-2\pi i y\cdot\xi}\,dy \\ &+ \int_{\substack{\{\max\{r,|\xi|^{-1}\}<|y|<s\} \\ \backslash\{\max\{r,|\xi|^{-1}\}<|y-\frac{\xi}{2|\xi|^2}|<s\}}} K(y - \frac{\xi}{2|\xi|^2})e^{-2\pi i y\cdot\xi}\,dy\end{aligned}$$

Then we have

$$\begin{aligned}&\Big|\int_{\max\{r,|\xi|^{-1}\}<|y|<s}[K(y) - K(y - \frac{\xi}{2|\xi|^2})]e^{-2\pi i y\cdot\xi}\,dy\Big| \\ &\leq \int_{|\xi|^{-1}<|y|}\big|K(y) - K(y - \frac{\xi}{2|\xi|^2})\big|\,dy \leq B\end{aligned}$$

uniformly in $\xi \neq 0$. For the second integral above over the more complicated region

$$\{\max\{r, |\xi|^{-1}\} < |y - \frac{\xi}{2|\xi|^2}| < s\} \backslash \{\max\{r, |\xi|^{-1}\} < |y| < s\}$$

note that we either have

$$|y| \geq s, \, \max\{r, |\xi|^{-1}\} < |y - \frac{\xi}{2|\xi|^2}| < s$$

which implies

$$|y - \frac{\xi}{2|\xi|^2}| \geq -\frac{1}{2}|\xi|^{-1} + s \geq \frac{1}{2}s,$$

or else

$$|y| < \max\{r, |\xi|^{-1}\}, \, \max\{r, |\xi|^{-1}\} < |y - \frac{\xi}{2|\xi|^2}| < s,$$

which gives

$$|y - \frac{\xi}{2|\xi|^2}| < \max\{r, |\xi|^{-1}\} + \frac{1}{2}|\xi|^{-1} \leq \frac{3}{2} \max\{r, |\xi|^{-1}\}.$$

In either case there is a number $\gamma > 0$ such that $|y - \frac{\xi}{2|\xi|^2}| \in [\gamma, 2\gamma]$. But then, using condition (1) on the kernel, we get

$$\begin{aligned}
&\Big| \int_{\substack{\max\{r,|\xi|^{-1}\}<|y-\frac{\xi}{2|\xi|^2}|<s \\ \backslash\{\max\{r,|\xi|^{-1}\}<|y|<s\}}} K(y - \frac{\xi}{2|\xi|^2}) e^{-2\pi i y \cdot \xi} \, dy \Big| \\
&\leq \int_{|y-\frac{\xi}{2|\xi|^2}|\in[\gamma,2\gamma]} \big| K(y - \frac{\xi}{2|\xi|^2}) \big| \, dy \\
&\leq B \int_{|y-\frac{\xi}{2|\xi|^2}|\in[\gamma,2\gamma]} |y - \frac{\xi}{2|\xi|^2}|^{-n} \, dy \\
&\leq B_1
\end{aligned}$$

The integral

$$\int_{\substack{\{\max\{r,|\xi|^{-1}\}<|y|<s\} \\ \backslash\{\max\{r,|\xi|^{-1}\}<|y-\frac{\xi}{2|\xi|^2}|<s\}}} K(y - \frac{\xi}{2|\xi|^2}) e^{-2\pi i y \cdot \xi} \, dy$$

is similar.

□