Serie 3
Optimal transport, Fall semester
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Exercise 3.1. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a nonnegative lower semicontinuous function. Show that:

(i) f admits a minimizer in every compact set $K \subset \mathbb{R}^n$.

(ii) f can be approximated from below monotonically (namely, $f_\lambda(x) \uparrow f(x)$ for every $x \in \mathbb{R}^d$) by a sequence of functions f_λ as $\lambda \to \infty$, where f_λ is λ-Lipschitz.

(iii) for every sequence of measures $\mu_n \rightharpoonup \mu$ narrowly,

$$\liminf_{n\to\infty} \int_{\mathbb{R}^d} f \, d\mu_n \geq \int_{\mathbb{R}^d} f \, d\mu.$$

Hint: For (ii), define $f_\lambda(x) = \inf_{y\in\mathbb{R}^n} \{f(y) + \lambda|x-y|\}$.

Solution:

(i) Let $K \subset \mathbb{R}^d$ be a compact set and $\{x_n\}_{n=1}^\infty \subset K$ a minimizing sequence, i.e.

$$\lim_{n\to\infty} f(x_n) = \inf\{f(x) : x \in K\}.$$

Since K is compact, there is a converging subsequence $\{x_{n_k}\}_{k=1}^\infty$ such that $x_{n_k} \to x \in K$. Then, since f is lower semicontinuous,

$$f(x) \leq \liminf_{k\to\infty} f(x_{n_k}) = \inf\{f(x) : x \in K\}.$$

Thus

$$f(x) \leq f(y) \quad \forall y \in K.$$

(ii) Define $f_\lambda(x) = \inf_{y\in\mathbb{R}^n} \{f(y) + \lambda|x-y|\}$. We begin by showing that $\{f_\lambda\}_{\lambda\in\mathbb{R}}$ is increasing in λ and bounded by f. Assume $\lambda < \lambda'$, then

$$f_\lambda(x) \leq f(y) + \lambda|x-y| \leq f(y) + \lambda'|x-y| \quad \text{for every } y \in \mathbb{R}^d.$$

Taking the infimum over y, we get $f_\lambda(x) \leq f_{\lambda'}(x)$. In addition due to the definition of f_λ, we get $f_\lambda(x) \leq f(x)$. Now we prove that f_λ is indeed λ-Lipschitz. We have

$$f_\lambda(x') \leq f(y) + \lambda|x'-y| \leq f(y) + \lambda|x-y| + \lambda|x'-x|.$$

Taking the infimum over y, we get $f_\lambda(x') \leq f_\lambda(x) + \lambda|x'-x|$. In a similar way, we can prove $f_\lambda(x) \leq f_\lambda(x') + \lambda|x'-x|$. Hence, $|f_\lambda(x') - f_\lambda(x)| \leq \lambda|x-x'|$, for all $x, x' \in \mathbb{R}^d$, proving that f_λ is λ-Lipschitz.

Now finally, we prove that $\lim_{\lambda\to\infty} f_\lambda = f$. Let $x \in \mathbb{R}^d$. Since f is lower semicontinuous, for any $\epsilon > 0$, there is $\delta > 0$ so that for all y such that $|x-y| \le \delta$ we have $f(y) \ge \min\{f(x)-\epsilon, \frac{1}{\epsilon}\}$. In addition, since f is nonnegative $f(y) + \lambda|x-y| \ge \delta\lambda$, for all y such that $|x-y| > \delta$. We conclude

$$f_\lambda(x) \ge \min\{f(x) - \epsilon, \frac{1}{\epsilon}, \delta\lambda\}.$$

Letting $\lambda \to \infty$, we get

$$\liminf_{\lambda\to\infty} f_\lambda(x) \ge \min\{f(x) - \epsilon, \frac{1}{\epsilon}\}.$$

Since ϵ is arbitrary, we deduce $\liminf_{\lambda\to\infty} f_\lambda(x) \ge f(x)$. Finally, since f_λ is bounded by f, we deduce

$$\lim_{\lambda\to\infty} f_\lambda(x) = f(x).$$

(iii) Notice that the function $\min\{f_\lambda, \lambda\}$ is continuous and bounded so that since $\mu_n \rightharpoonup \mu$,

$$\int_{\mathbb{R}^d} \min\{f_\lambda, \lambda\}\, d\mu = \lim_{n\to\infty} \int_{\mathbb{R}^d} \min\{f_\lambda, \lambda\}\, d\mu_n \le \liminf_{n\to\infty} \int_{\mathbb{R}^d} f\, d\mu_n,$$

where the inequality follows from the fact that $f_\lambda \le f$. Letting $\lambda \to \infty$, we get, due to the monotone convergence theorem,

$$\int_{\mathbb{R}^d} f\, d\mu \le \liminf_{n\to\infty} \int_{\mathbb{R}^d} f\, d\mu_n.$$

Exercise 3.2. The support of a nonnegative measure $\mu \in \mathcal{M}_+(\mathbb{R}^n)$ is defined as the smallest closed set on which μ is concentrated, i.e.

$$spt(\mu) := \bigcap\{C \subset \mathbb{R}^n \text{ closed } : \mu(\mathbb{R}^n \setminus C) = 0\}.$$

Let us take a sequence of nonnegative measures $\mu_j \in \mathcal{M}_+(\mathbb{R}^n)$ such that $\mu_j \overset{*}{\rightharpoonup} \mu$. Prove the following fact: for every $x \in spt(\mu)$, there exists a sequence of points $x_j \in spt(\mu_j)$ such that $x_j \to x$.

Solution: Suppose by contradiction that for some $\epsilon > 0$ there is a sequence $j_k \to \infty$ for which $spt(\mu_{j_k}) \subset \mathbb{R}^n \setminus B_\epsilon(x)$. Let us consider a test function $\varphi \in C_c(B_\epsilon(x))$ such that

$$\begin{cases} \varphi \ge 0 & \text{in } \mathbb{R}^n, \\ \varphi = 0 & \text{in } \mathbb{R}^n \setminus B_\epsilon(x), \\ \varphi = 1 & \text{in } B_{\epsilon/2}(x). \end{cases}$$

Since $x \in spt(\mu)$, $\mu(B_{\epsilon/2}(x)) > 0$. Testing the weak* convergence of μ_{j_k} to μ with φ, we get

$$0 = \int \varphi d\mu_{j_k} \overset{k\to\infty}{\longrightarrow} \int \varphi d\mu \ge \mu(B_{\epsilon/2}(x)) > 0,$$

a contradiction.

Exercise 3.3 (Characterizations of weak-$*$ convergence)**.** Let $\mu_j, \mu \in \mathscr{P}(\mathbb{R}^n)$ be probability measures in $\mathbb{R}^n$.

i) Show that $\mu_j \rightharpoonup \mu$ narrowly if and only if one of the following properties hold:

   a) For every open set $A \subset \mathbb{R}^n$:
$$\liminf_{j\to\infty} \mu_j(A) \geq \mu(A).$$

   b) For every closed set $C \subset \mathbb{R}^n$:
$$\limsup_{j\to\infty} \mu_j(C) \leq \mu(C).$$

   c) For every set $E \subset \mathbb{R}^n$ such that $\mu(\partial E) = 0$:
$$\lim_{j\to\infty} \mu_j(E) = \mu(E).$$

ii) Give an example of a sequence of probability measures $\mu_j \in \mathscr{P}(\mathbb{R}^n)$ such that $\mu_j \overset{*}{\rightharpoonup} \mu$ for some measure $\mu \in \mathcal{M}_+(\mathbb{R}^n)$ and an open set A such that
$$\liminf_{j\to\infty} \mu_j(A) > \mu(A).$$

Hint: For one implication in (i), use Exercise 3.1. For the other, use the layer-cake formula
$$\int \varphi d\mu = \int_0^\infty \mu(\{\varphi > t\})dt \quad \text{for every } \varphi \in C_b(\mathbb{R}^n),\ \varphi \geq 0.$$

Solution:

(i) We will prove the following chain of implications:
$$\mu_j \rightharpoonup \mu \text{ narrowly} \implies a) \implies b) \implies c) \implies \mu_j \rightharpoonup \mu \text{ narrowly}.$$

$\mu_j \rightharpoonup \mu$ narrowly $\implies$ *a*). Let $\mu_j \rightharpoonup \mu$ narrowly and take A an open set. Notice that the function $\mathbb{1}_A$ is lower semicontinuous since A is open. Thus, using Exercise 3.1
$$\liminf_{j\to\infty} \mu_j(A) = \liminf_{j\to\infty} \int \mathbb{1}_A \, d\mu_n \geq \int \mathbb{1}_A \, d\mu = \mu(A).$$

a) $\implies$ *b*). Now assume *a*) holds true and take C a closed set. Then $A := \mathbb{R}^n \setminus C$ is open, therefore, by *a*):
$$\begin{aligned}\limsup_{j\to\infty} \mu_j(C) &= \limsup_{j\to\infty} (1 - \mu_j(A)) = 1 - \liminf_{j\to\infty} \mu_j(A) \\ &\leq 1 - \mu(A) = \mu(C).\end{aligned}$$

b) $\implies$ *c*). Assume *b*) holds true. With the same argument as above we can easily show that *a*) holds true as well. Let $E \subset \mathbb{R}^n$ be a set so that $\mu(\partial E) = 0$. We call $A := E \setminus \partial E$

and $C := \overline{E}$, and note that A is open and C is closed. Then, using $a)$, $b)$ and $\mu(\partial E) = 0$ we obtain the following chain of inequalities:

$$\limsup_{j\to\infty} \mu_j(E) \le \limsup_{j\to\infty} \mu_j(C) \le \mu(C) = \mu(E) =$$
$$= \mu(A) \le \liminf_{j\to\infty} \mu_j(A) \le \liminf_{j\to\infty} \mu_j(E).$$

Hence the previous inequalities must be all equalities proving that $\mu_j(E) \to \mu(E)$, as desired.

$c) \implies \mu_j \rightharpoonup \mu$ narrowly. Assume that $c)$ holds true and let $\varphi \in C_b(\mathbb{R}^n)$ be a nonnegative bounded continuous function. Observe that for almost every $t > 0$, the set $E_t := \{\varphi > t\}$ is such that $\mu(\partial E_t) = 0$. Hence, by $c)$:

$$\lim_{j\to\infty} \mu_j(E_t) = \mu(E_t) \quad \text{for almost every } t > 0.$$

Therefore, using the layer cake formula, together with the dominated convergence theorem we get

$$\int \varphi d\mu = \int_0^{\max\varphi} \mu(E_t)dt = \lim_{j\to\infty} \int_0^{\max\varphi} \mu_j(E_t)dt = \lim_{j\to\infty} \int \varphi d\mu_j.$$

This proves that $\mu_j \rightharpoonup \mu$ narrowly.

(ii) Let $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}^n$ be a sequence of points such that $|x_j| \to \infty$ as $j \to \infty$. For any $j \in \mathbb{N}$, we define $\mu_j = \delta_{x_j}$. Now for any $\varphi \in C_c(\mathbb{R}^n)$,

$$\lim_{j\to\infty} \int \varphi \, d\mu_j = \lim_{j\to\infty} \varphi(x_j) = 0 \quad \text{since} \quad |x_j| \to \infty.$$

Hence, $\mu_j \overset{*}{\rightharpoonup} \mu$, where μ is the null measure. Now taking $A = \mathbb{R}^n$ we get

$$1 = \liminf_{j\to\infty} \mu_j(\mathbb{R}^n) > \mu(\mathbb{R}^n) = 0.$$

Exercise 3.4. Let $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathscr{P}(\mathbb{R})$ be a sequence of probability measures with $\mu_n \rightharpoonup \mu$ narrowly. Define $F_n(x) := \mu_n((-\infty, x])$, $F(x) := \mu((-\infty, x])$.

(i) Prove that μ is a probability measure.

(ii) Prove that

$$\limsup_n F_n(x) \le F(x) \quad \text{for every } x \in \mathbb{R}.$$

(iii) Prove that

$$\lim_n F_n(x) = F(x) \quad \text{for every } x \in \mathbb{R} \text{ at which } F \text{ is continuous.}$$

(iv) Give an example of a sequence of measures $\mu_n \rightharpoonup \mu$ narrowly and an $x \in \mathbb{R}$ for which

$$\limsup_n F_n(x) < F(x).$$

Solution:

(i) To prove that μ is a probability measure, use the definition of narrow convergence with the test function $f \equiv 1$.

(ii) We have the following

$$\begin{aligned}\limsup_n F_n(x) &= \limsup_n (1 - \mu_n((x, \infty))) \\ &= 1 - \liminf_n \mu_n((x, \infty)) \\ &\leq 1 - \mu((x, \infty)) = F(x),\end{aligned}$$

which gives the thesis.

(iii) Let us fix a point $x \in \mathbb{R}$ and $\delta > 0$: we take a continuous non increasing function f such that $f(t) \equiv 1$ for $t \leq x$ and $f(t) \equiv 0$ for every $t \geq x + \delta$, then we have

$$F(x + \delta) \geq \int_{\mathbb{R}} f(x) d\mu(x) = \lim_n \int_{\mathbb{R}} f(x) d\mu_n(x) \geq \limsup_n F_n(x).$$

Now consider a non increasing function f such that $f(t) \equiv 1$ for $t \leq x - \delta$ and $f(t) \equiv 0$ for every $t \geq x$, then we have

$$F(x - \delta) \leq \int_{\mathbb{R}} f(x) d\mu(x) = \lim_n \int_{\mathbb{R}} f(x) d\mu_n(x) \leq \liminf_n F_n(x),$$

the thesis follows from the continuity of F in x.

(iv) For the example consider $\mu_n = \delta_{1/n}$, $\mu = \delta_0$ and $x = 0$.

Exercise 3.5.

(i) Find a sequence of functions $f_n : [0, 1] \to [0, 1]$ such that $(f_n)_\#(\mathscr{L}^1 \llcorner [0, 1]) = (\mathscr{L}^1 \llcorner [0, 1])$ but f_n weakly converge to $1/2$.

(ii) What is the weak limit of $(\mathrm{id}, f_n)_\# \mathscr{L}^1 \llcorner [0, 1]$?

(iii) (💣) Can these functions be taken C^1?

Hint: For (i), use piecewise affine oscillating functions.

Solution:

(i) Define $\phi \colon \mathbb{R} \to \mathbb{R}$ on $[0, 1]$ by

$$\phi(x) = \begin{cases} 2x & \text{if } x \in [0, 1/2]; \\ 2(1 - x) & \text{if } x \in (1/2, 1], \end{cases}$$

and extend it to $\mathbb{R}$ by 1-periodicity. Define $f_n\colon [0,1] \to [0,1]$ by

$$f_n(x) = \phi(nx).$$

From the Riemann-Lebesgue theorem, it is clear that f_n converges weakly to $1/2$. In order to prove $(f_n)_\#(\mathscr{L}|_{[0,1]}) = (\mathscr{L}|_{[0,1]})$ we need to show that

$$(\mathscr{L}|_{[0,1]})(f_n^{-1}(A)) = (\mathscr{L}|_{[0,1]})(A) \text{ for any } A \subseteq \mathbb{R} \text{ Borel set.} \tag{1}$$

Since the Borel σ-algebra is generated by sets of the form $A = (a, \infty)$ it suffices to show the previous equality for sets of this form. First we have

$$\phi^{-1}((a,\infty)) = \bigcup_{m\in\mathbb{Z}} (m + a/2, m + 1 - a/2) \quad \text{if } a \in [0,1).$$

If $a \geq 1$, then $\phi^{-1}((a,\infty)) = \emptyset$ and if $a < 0$, then $\phi^{-1}((a,\infty)) = \mathbb{R}$. From this we deduce that

$$f_n^{-1}((a,\infty)) = \bigcup_{m=0}^{n-1} \left(\frac{m}{n} + \frac{a}{2n}, \frac{m+1}{n} - \frac{a}{2n} \right) \quad \text{if } a \in [0,1).$$

In addition, $f_n^{-1}((a,\infty)) = \emptyset$ if $a \geq 1$ and $f_n^{-1}((a,\infty)) = [0,1]$ if $a < 0$. Thus the equation (1) is satisfied if $a \geq 1$ or $a < 0$. Now considering the case when $a \in [0,1)$, we have

$$(\mathscr{L}|_{[0,1]})(f_n^{-1}((a,\infty))) = (\mathscr{L}|_{[0,1]}) \left(\bigcup_{m=0}^{n-1} \left(\frac{m}{n} + \frac{a}{2n}, \frac{m+1}{n} - \frac{a}{2n} \right) \right) = \sum_{m=0}^{n-1} \frac{1-a}{n} = 1 - a.$$

which implies (1). In conclusion, since (1) holds for any set of the form $A = (a,\infty)$ and the family of these sets generate the Borel σ-algebra, (1) holds for any Borel set which proves that

$$(f_n)_\#(\mathscr{L}|_{[0,1]}) = (\mathscr{L}|_{[0,1]}).$$

(ii) Let $\varphi \in C_c(\mathbb{R} \times \mathbb{R})$. We will show

$$\int \varphi(x,y)\, d(\mathrm{id}, f_n)_\#\mathscr{L}|_{[0,1]} \to \int \varphi(x,y)\, d\mathscr{L}^2|_{[0,1]^2} \quad \text{as} \quad n \to \infty.$$

Fix any $\epsilon > 0$. Since φ is uniformly continuous there is $\delta > 0$ such that if $|x - y| < \delta$, then $|\varphi(x) - \varphi(y)| < \epsilon$. Let $n \in \mathbb{N}$ be large enough so that $2/n < \delta$. Now we will consider a grid covering $[0,1]^2$ composed by $2n^2$ rectangles of the form

$$R_{ij} = \left[\frac{i}{2n}, \frac{i+1}{2n} \right] \times \left[\frac{j}{n}, \frac{j+1}{n} \right] \quad i = 0, \dots, 2n-1, j = 0, \dots, n-1.$$

Notice that all R_{ij} have a diameter smaller than δ. Now for any $k = 0, \dots, 2n^2 - 1$, denote i_k, j_k the unique integers such that

$$\left(\frac{2k+1}{4n^2}, f_n\left(\frac{2k+1}{4n^2} \right) \right) \in R_{i_k j_k}$$

Then, using the fact that φ is uniformly continuous, we get

$$\left|\int_{\frac{k}{2n^2}}^{\frac{k+1}{2n^2}} \varphi(x, f_n(x))\, dx - \frac{1}{2n^2}\varphi\left(\frac{2k+1}{4n^2}, f_n\left(\frac{2k+1}{4n^2}\right)\right)\right| < \frac{1}{2n^2}\epsilon \quad \forall k = 1, \ldots, 2n^2 - 1$$

and

$$\left|\int_{R_{i_k j_k}} \varphi(x, y)\, dxdy - \frac{1}{2n^2}\varphi\left(\frac{2k+1}{4n^2}, f_n\left(\frac{2k+1}{4n^2}\right)\right)\right| < \frac{1}{2n^2}\epsilon \quad \forall k = 1, \ldots, 2n^2 - 1.$$

Therefore, we get

$$\left|\int \varphi(x, y)\, d(\mathrm{id}, f_n)_\# \mathscr{L}|_{[0,1]} - \frac{1}{2n^2}\sum_{k=1}^{2n^2-1} \varphi\left(\frac{2k+1}{4n^2}, f_n\left(\frac{2k+1}{4n^2}\right)\right)\right|$$
$$\leq \frac{1}{2n^2}\sum_{k=1}^{2n^2-1}\left|\int_{\frac{k}{2n^2}}^{\frac{k+1}{2n^2}} \varphi(x, f_n(x))\, dx - \varphi\left(\frac{2k+1}{4n^2}, f_n\left(\frac{2k+1}{4n^2}\right)\right)\right| < \epsilon$$

and

$$\left|\int \varphi(x, y)\, d\mathscr{L}|_{[0,1]^2} - \frac{1}{2n^2}\sum_{k=1}^{2n^2-1} \varphi\left(\frac{2k+1}{4n^2}, f_n\left(\frac{2k+1}{4n^2}\right)\right)\right|$$
$$\leq \frac{1}{2n^2}\sum_{k=1}^{2n^2-1}\left|\int_{R_{i_k j_k}} \varphi(x, y)\, dxdy - \varphi\left(\frac{2k+1}{4n^2}, f_n\left(\frac{2k+1}{4n^2}\right)\right)\right| < \epsilon.$$

This proves that

$$\left|\int \varphi(x, y)\, d(\mathrm{id}, f_n)_\# \mathscr{L}|_{[0,1]} - \int \varphi(x, y)\, d\mathscr{L}^2|_{[0,1]^2}\right| < 2\epsilon$$

when n is large enough. We conclude that $(\mathrm{id}, f_n)_\# \mathscr{L}|_{[0,1]}$ converges weakly to $\mathscr{L}^2|_{[0,1]^2}$.

(iii) Assume $\varphi \in C^1(\mathbb{R})$ such that $(\varphi)_\#(\mathscr{L}|_{[0,1]}) = \mathscr{L}|_{[0,1]}$. For a contradiction, assume there is $y \in (0, 1)$ such that $\varphi'(y) = 0$. For any $\epsilon > 0$, there is $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x - y| < \delta$ we have $|\varphi'(x)| < \epsilon$. Thus, for all x such that $|x - y| < \delta$, we get

$$|\varphi(x) - \varphi(y)| < \epsilon|x - y| < \epsilon\delta$$

Then

$$(x - \delta, x + \delta) \subseteq \varphi^{-1}((\varphi(x) - \epsilon\delta, \varphi(x) + \epsilon\delta)),$$

so that

$$2\epsilon\delta = (\varphi)_\#(\mathscr{L}|_{[0,1]})((\varphi(x) - \epsilon\delta, \varphi(x) + \epsilon\delta)) = (\mathscr{L}|_{[0,1]})(\varphi^{-1}((\varphi(x) - \epsilon\delta, \varphi(x) + \epsilon\delta))) \geq 2\delta$$

which yields a contradiction when taking $\epsilon < 1$. This proves that φ' never vanishes on $(0, 1)$ so that φ is either strictly increasing or strictly decreasing. Assume φ is stricly increasing (the other case is similar). Then for any $a, b \in (0,1)$, we have

$$b - a = (\varphi)_{\#}(\mathscr{L}|_{[0,1]})((a,b)) = \mathscr{L}|_{[0,1]}((\varphi^{-1}(a), \varphi^{-1}(b))) = \varphi^{-1}(b) - \varphi^{-1}(a).$$

Thus, for any $x, y \in (0,1)$, $\varphi(x) - \varphi(y) = x - y$. We conclude $\varphi(x) = x$ on $(0,1)$. If we assume φ is strictly decreasing, then $\varphi(x) = 1 - x$ on $(0,1)$. We conclude by noting that any sequence of such functions cannot converge weakly to $1/2$.

Exercise 3.6 (💣). Let $\mu \in \mathscr{P}(\mathbb{R}^n)$ be a probability measure. We say that a sequence of borel functions $T_j : \mathbb{R}^n \to \mathbb{R}^n$ converge in μ-measure to $T : \mathbb{R}^n \to \mathbb{R}^n$ if[1]

$$\lim_{j\to\infty} \mu\left(\{x \in \mathbb{R}^n : |T_j(x) - T(x)| > \epsilon\}\right) = 0 \quad \text{for every } \epsilon > 0.$$

Denoting by $\pi_j := (id, T_j)_{\#}\mu, \pi := (id, T)_{\#}\mu \in \mathscr{P}(\mathbb{R}^n \times \mathbb{R}^n)$, prove the following equivalence:

$$\pi_j \overset{*}{\rightharpoonup} \pi \qquad \Longleftrightarrow \qquad T_j \text{ converges to } T \text{ in } \mu\text{-measure}$$

Solution: Assume first that T_j converges to T in μ-measure. We wish to prove that $(id, T_j)_{\#}\mu$ weakly-$*$ converge to $(id, T)_{\#}\mu$. Let $\varphi \in C_c(\mathbb{R}^n \times \mathbb{R}^n)$. Then, from each subsequence $j_k \nearrow \infty$, we can extract a further subsequence $j_{k_\ell} \nearrow \infty$ for which $T_{j_{k_\ell}}$ converges to T pointwise μ-almost everywhere, and so, by the change of variables formula and the dominated convergence theorem we obtain

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \varphi d(id, T_{j_{k_\ell}})_{\#} d\mu = \int_{\mathbb{R}^n} \varphi(x, T_{j_{k_\ell}}(x)) d\mu \overset{\ell\to\infty}{\to} \int_{\mathbb{R}^n} \varphi(x, T(x)) d\mu = \int_{\mathbb{R}^n\times\mathbb{R}^n} \varphi d(id, T)_{\#} d\mu,$$

which gives the desired result.

Let us now assume that $\pi_j = (id, T_j)_{\#}\mu$ weakly-$*$ converge to $\pi = (id, T)_{\#}\mu$ (and hence narrowly, as all these measures have the same total mass), and prove that T_j converges to T in μ-measure. Fix $\epsilon > 0$. By Lusin's Theorem, there exists a continuous map $\tilde{T} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mu(\{\tilde{T} \neq T\}) < \epsilon$. Let us then consider the continuous bounded test function $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined as

$$\varphi(x, y) := \min\{|y - \tilde{T}(x)|, 1\}.$$

By the change of variables formula and the narrow convergence,

$$\begin{aligned}\int_{\mathbb{R}^n} \min\{|T_j(x) - \tilde{T}(x)|, 1\} d\mu = \int_{\mathbb{R}^n\times\mathbb{R}^n} \varphi d\pi_j &\to \int_{\mathbb{R}^n\times\mathbb{R}^n} \varphi d\pi \\ &= \int_{\mathbb{R}^n} \min\{|T(x) - \tilde{T}(x)|, 1\} d\mu \le \mu(\{T \neq \tilde{T}\}) < \epsilon.\end{aligned}$$

[1]By standard measure theory arguments one can easily prove that whenever T_j converges to T in μ-measure, there is a subsequence $j_k \nearrow \infty$ such that T_{j_k} converge to T pointwise μ-almost everywhere.

Thus if j is large enough we have

$$\int_{\mathbb{R}^n} \min\{|T_j(x) - T(x)|, 1\} d\mu \leq \int_{\mathbb{R}^n} \min\{|T_j(x) - \tilde{T}(x)|, 1\} d\mu + \int_{\mathbb{R}^n} \min\{|T(x) - \tilde{T}(x)|, 1\} d\mu < 2\epsilon.$$

Finally, given any $\delta > 0$, using Markov's inequality, we get, for j large enough:

$$\mu(\{|T_j - T| > \delta\}) \leq \frac{1}{\delta} \int_{\mathbb{R}^n} \min\{|T_j(x) - T(x)|, 1\} d\mu \leq \frac{2\epsilon}{\delta},$$

and the thesis follows from the arbitrariness of ϵ.