Serie 2

Optimal transport, Fall semester

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Exercise 2.1. Determine the measure $T_{\#}\mu$ for the map $T = \nabla\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, where the function $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\varphi(x_1, x_2) = \frac{1}{3}(x_1^3 + x_2^3)$, for the following choices of μ:

(i) $\mu = \delta_{(0,0)} + \delta_{(1,2)}$;

(ii) $\mu = \mathscr{L}^2 \llcorner [0,1]^2$.

Consider the Monge problem with the cost function $c(x, y) = \frac{|x-y|^2}{2}$. Is the map T optimal from μ to $T_{\#}\mu$, for μ as in (i)?

Solution:

(i) Note that $T(x_1, x_2) = (x_1^2, x_2^2)$. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a bounded Borel function, then

$$\begin{aligned}\int_{\mathbb{R}^2} \varphi \, d(T_{\#}\mu) &= \int_{\mathbb{R}^2} \varphi \circ T \, d\mu = \varphi(T(0,0)) + \varphi(T(1,2)) \\ &= \varphi(0,0) + \varphi(1,4) = \int_{\mathbb{R}^2} \varphi \, d(\delta_{(0,0)} + \delta_{(1,4)}).\end{aligned}$$

Thus, $T_{\#}\mu = \delta_{(0,0)} + \delta_{(1,4)}$.

(ii) In a similar way, with $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ a bounded Borel function,

$$\begin{aligned}\int_{\mathbb{R}^2} \varphi \, d(T_{\#}(\mathbb{1}_{[0,1]^2}\mathscr{L}^2)) &= \int_{\mathbb{R}^2} \varphi \circ T \, d(\mathbb{1}_{[0,1]^2}\mathscr{L}^2) \\ &= \int_{[0,1]^2} \varphi(T(x_1, x_2)) \, d\mathscr{L}^2(x_1, x_2) \\ &= \int_{[0,1]^2} \varphi(T(\sqrt{u}, \sqrt{v})) \left| \det \begin{pmatrix} \frac{1}{2\sqrt{u}} & 0 \\ 0 & \frac{1}{2\sqrt{v}} \end{pmatrix} \right| d\mathscr{L}^2(u, v) \\ &= \int_{[0,1]^2} \varphi(u, v) \frac{1}{4\sqrt{uv}} d\mathscr{L}^2(u, v),\end{aligned}$$

where we used the change of variables formula. Thus, $T_{\#}\mu = \mathbb{1}_{[0,1]^2} \dfrac{1}{4\sqrt{xy}} \mathscr{L}^2$.

(iii) Consider the case $\mu = \delta_{(0,0)} + \delta_{(1,2)}$. From the first point we have that $T_{\#}\mu = \delta_{(0,0)} + \delta_{(1,4)}$. For the Monge problem each competitor $\tilde{T} : \mathbb{R}^2 \to \mathbb{R}$ is such that $\tilde{T}_{\#}\mu = \delta_{(0,0)} + \delta_{(1,4)}$ which implies that

$$\begin{cases} \tilde{T}((0,0)) = (0,0), \\ \tilde{T}((1,2)) = (1,4). \end{cases} \quad \text{or} \quad \begin{cases} \tilde{T}((0,0)) = (1,4), \\ \tilde{T}((1,2)) = (0,0). \end{cases}$$

In the first case, the cost of $\tilde{T}$ is clearly the same as the cost of T, because T and $\tilde{T}$ coincide in the support of μ. In the second case we can explicitly compute

$$\int_{\mathbb{R}^2} c(x, \tilde{T}(x))d\mu = \frac{|(1,4)-(0,0)|^2}{2} + \frac{|(0,0)-(1,2)|^2}{2} = 11 > 2 = \int_{\mathbb{R}^2} c(x, T(x))d\mu.$$

Hence T is optimal for the Monge problem.

Exercise 2.2. Let us introduce the following notation for the Monge problem:

$$MP(\mu, \nu, c) := \inf \left\{ \int_X c(x, T(x))d\mu(x) : T_\# \mu = \nu \right\}.$$

(i) Give a cost function $c : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ of the form $c(x, y) = f(|x - y|)$ for some Lipschitz function $f : [0, \infty) \to [0, \infty)$ and a measure μ for which the identity is not an optimal map for the problem $MP(\mu, \mu, c)$. More precisely, give a cost function c as above, a measure μ and a map T such that $T_\# \mu = \mu$ and

$$\int_{\mathbb{R}^2} c(x, T(x))d\mu(x) < \int_{\mathbb{R}^2} c(x, x)d\mu(x).$$

(ii) Let $c : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ be a measurable cost function such that $c(x, y) = f(|x - y|)$ for a measurable $f : [0, \infty) \to [0, \infty)$. What is a necessary and sufficient condition on the cost function c in such a way that the identity is an optimal map for $MP(c, \mu, \mu)$ for any $\mu \in \mathscr{P}(\mathbb{R}^n)$?

Solution:

(i) Consider $c(x, y) = \max\{0, \frac{1}{2} - |x - y|\}$, $\mu = \mathbb{1}_{B(0,1) \setminus B(0,1/2)}$ and $T(x) = R_\pi(x)$, where R_π is the rotation map of an angle π namely

$$R_\pi = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

We let the reader verify that $(R_\pi)_\# \mu = \mu$ and

$$0 = \int_{\mathbb{R}^2} c(x, R_\pi(x))d\mu(x) < \int_{\mathbb{R}^2} c(x, x)d\mu(x) = \frac{1}{2}.$$

(ii) We claim that the necessary and sufficient condition on $c(x, y) = f(|x - y|)$ is that

$$f(0) = \min_{x \in [0, \infty)} f(x).$$

It is a sufficient condition: we have that

$$\int_{\mathbb{R}^n} c(x, x)d\mu(x) = \int_{\mathbb{R}^n} f(0)d\mu(x) \leq \int_{\mathbb{R}^n} f(|x - T(x)|)d\mu(x),$$

for any $\mu \in \mathscr{P}(\mathbb{R}^n)$ and any map $T : \mathbb{R}^n \to \mathbb{R}^n$.

It is a necessary condition: suppose by contradiction that the identity is an optimal map of the problem $MP(\mu, \mu, c)$ for any $\mu \in \mathscr{P}(\mathbb{R}^n)$ and there exists $y > 0$ such that $f(y) < f(0)$, then we define $\mu = \frac{1}{2}\delta_{\underline{0}} + \frac{1}{2}\delta_{ye_1}$, where e_1 is the first vector of the standard basis of $\mathbb{R}^n$. Fix a map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(0) = ye_1$ and $T(ye_1) = 0$. Such a map satisfies $T_{\#}\mu = \mu$ and

$$\int_{\mathbb{R}^n} c(x, T(x))d\mu(x) < \int_{\mathbb{R}^n} c(x, x)d\mu(x)$$

which contradicts the minimality of the identity map.

Exercise 2.3. Find the unique monotone map $T : [0, 1] \to [0, \infty)$ such that $T_{\#}\mu = \nu$, where the measures $\mu, \nu \in \mathscr{P}(\mathbb{R})$ are given by $\mu = \mathscr{L}^1 \llcorner [0, 1]$ and $\nu = e^{-x}\mathscr{L}^1 \llcorner [0, \infty)$.

Solution: We are looking for the transport map between two absolutely continuous measures on the real line. In particular, $T : [0, 1] \to [0, \infty)$ is a monotone increasing function. This, coupled with the transport condition forces

$$\mu([0, x]) = \nu(T([0, x])) = \nu([0, T(x)])$$

for all $x \in [0, 1]$. That is,

$$x = \int_0^{T(x)} e^{-t}\, dt = 1 - e^{-T(x)} \quad \Leftrightarrow \quad T(x) = -\log(1 - x).$$

So the map we are looking for is $T(x) = -\log(1-x)$. Alternatively, one can directly use monotone rearrangement (see Section 1.4.1) to deduce the desired result.

The next three exercises are devoted to the derivation of some properties of the Knothe transport map. Given $f, g : \mathbb{R}^2 \to \mathbb{R}$ two positive functions with integral 1, we define

$$F(x_1) = \int_{\mathbb{R}} f(x_1, x_2)\, dx_2, \qquad G(x_1) = \int_{\mathbb{R}} g(x_1, x_2)\, dx_2.$$

Let $T_1 : \mathbb{R} \to \mathbb{R}$ be the monotone map which sends $F(x_1)\, dx_1$ to $G(x_1)\, dx_1$, namely $(T_1)_{\#}(F(x_1)\, dx_1) = G(x_1)\, dx_1$. For every $x_1 \in \mathbb{R}$, let $T_2(x_1, \cdot)$ be the monotone map which sends $f(x_1, x_2)/F(x_1)\, dx_2$ to $g(T_1(x_1), x_2)/G(T_1(x_1))\, dx_2$. The Knothe's map $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as

$$T(x_1, x_2) = (T_1(x_1), T_2(x_1, x_2)).$$

Exercise 2.4. Show that the Knothe's map $T(x_1, x_2) = (T_1(x_1), T_2(x_1, x_2))$ transports the measure $\mu = f(x_1, x_2)\, dx_1 dx_2$ to $\nu = g(x_1, x_2)\, dx_1 dx_2$.

Solution: For any $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ Borel and bounded, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} \varphi(y_1,y_2)\, d\nu(y_1,y_2) &= \int_{\mathbb{R}^2} \varphi(y_1,y_2) g(y_1,y_2)\, dy_1 dy_2 \\
&= \int_{\mathbb{R}} \underbrace{\left(\int_{\mathbb{R}} \varphi(y_1,y_2) \frac{g(y_1,y_2)}{G(y_1)}\, dy_2 \right)}_{:=\Psi(y_1)} G(y_1)\, dy_1 \\
&= \int_{\mathbb{R}} \Psi(y_1) G(y_1)\, dy_1 \\
&= \int_{\mathbb{R}} \Psi(T_1(x_1)) F(x_1)\, dx_1 \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi(T_1(x_1), y_2) \frac{g(T_1(x_1), y_2)}{G(T_1(x_1))}\, dy_2 \right) F(x_1)\, dx_1 \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varphi(T_1(x_1), T_2(x_1,x_2)) \frac{f(x_1,x_2)}{F(x_1)}\, dx_2 \right) F(x_1)\, dx_1 \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(T_1(x_1), T_2(x_1,x_2)) f(x_1,x_2)\, dx_2 dx_1 \\
&= \int_{\mathbb{R}} (\varphi \circ T)(x_1,x_2)\, d\mu(x_1,x_2),
\end{aligned}$$

where we used

$$(T_1)_\#(F(x_1)\, dx_1) = G(x_1)\, dx_1 \quad \text{and} \quad T_2(x_1,\cdot)_\# \left(\frac{f(x_1,\cdot)}{F(x_1)}\, dx_2 \right) = \frac{g(T_1(x_1),\cdot)}{G(T_1(x_1))}\, dy_2.$$

Exercise 2.5. Let T be a Knothe's map from $\mu = \frac{\mathbb{1}_E}{|E|}\, dx$ to $\nu = \frac{\mathbb{1}_{B_1}}{|B_1|}\, dy$, where $B_1 \subseteq \mathbb{R}^2$ is the unit ball and $E \subseteq \mathbb{R}^2$ a bounded open set with smooth boundary. Assuming that T is smooth, show that

(i) For any $x \in E$, it holds $|T(x)| \leq 1$.

(ii) $\det \nabla T = \frac{|B_1|}{|E|}$ in E.

(iii) $\operatorname{div} T \geq 2(\det \nabla T)^{\frac{1}{2}}$.

Hint: For (ii) and (iii), notice that the Jacobian ∇T of a Knothe's map T is upper triangular and all values on the diagonal are non-negative.

Solution:

(i) If $x \in E$, then $T(x) \in B_1$ and thus $|T(x)| \leq 1$.

(ii) Let $A \subseteq B_1$, so that $T^{-1}(A) \subseteq E$. Since $T_\#\mu = \nu$, we have

$$\nu(A) = \mu(T^{-1}(A)) = \int_{T^{-1}(A)} \frac{1}{|E|}\, dx.$$

On the other hand, using change of variables with $y = T(x)$ such that $dy = |\det \nabla T|\, dx$, we get

$$\nu(A) = \int_A \frac{1}{|B_1|}\, dy = \int_{T^{-1}(A)} \frac{1}{|B_1|} |\det \nabla T(x)|\, dx.$$

Since ∇T is upper triangular and its diagonal elements are nonnegative, $\det \nabla T \geq 0$, hence

$$\int_{T^{-1}(A)} \frac{1}{|E|}\, dx = \int_{T^{-1}(A)} \frac{1}{|B_1|} \det \nabla T(x)\, dx.$$

Since $A \subseteq B_1$ is arbitrary, we deduce

$$\frac{\det \nabla T}{|B_1|} = \frac{1}{|E|} \quad \text{inside } E.$$

(iii) First of all note that for any nonnegative numbers $x_1, \ldots, x_d$,

$$\left(\prod_{i=1}^d x_i\right)^{\frac{1}{d}} = \left(\prod_{i=1}^d \exp(\ln(x_i))\right)^{\frac{1}{d}} = \exp\left(\frac{1}{d}\sum_{i=1}^d \ln(x_i)\right) \leq \frac{1}{d}\sum_{i=1}^d \exp(\ln(x_i)) = \frac{1}{d}\sum_{i=1}^d x_i. \quad (1)$$

Since ∇T is upper-triangular, its determinant is given by the product of its diagonal elements. Hence

$$\operatorname{div} T(x) = \sum_{i=1}^2 \partial_i T_i(x) = 2\left(\frac{1}{2}\sum_{i=1}^2 \partial_i T_i(x)\right) \geq 2\left(\prod_{i=1}^2 \partial_i T_i(x)\right)^{\frac{1}{2}} = 2(\det \nabla T(x))^{\frac{1}{2}}$$

where the inequality follows from (1).

Exercise 2.6 (Isoperimetric inequality in $\mathbb{R}^2$)**.** Let $E \subset \mathbb{R}^2$ be a bounded open set with smooth boundary. Show that

$$\text{Length}(\partial E) \geq 2|B_1|^{\frac{1}{2}}|E|^{\frac{1}{2}}.$$

Solution: Denote by ν_E the outer unit normal to ∂E and by $d\sigma$ the surface measure on ∂E. Denote by T a Knothe's map from μ to ν as defined in Exercise 2.5. Note that $T \cdot \nu_E \leq |T||\nu_E| \leq |T| \leq 1$ due to (i) in Exercise 2.5. Therefore, we get

$$\begin{aligned}\text{Length}(\partial E) &= \int_{\partial E} 1\, d\sigma \geq \int_{\partial E} |T|\, d\sigma \geq \int_{\partial E} T \cdot \nu_E\, d\sigma \\ &= \int_E \operatorname{div} T\, dx \geq 2\int_E (\det \nabla T)^{\frac{1}{2}}\, dx = 2\int_E \left(\frac{|B_1|}{|E|}\right)^{\frac{1}{2}} dx = 2|B_1|^{\frac{1}{2}}|E|^{\frac{1}{2}},\end{aligned}$$

where we used (ii), (iii) from Exercise 2.5 and the divergence theorem.

Remark 2.1. The definition of the Knothe's map as well as the proof of the three previous exercises can be carried out in the same way in a general dimension d without many new ideas but at the price of a heavier notation. The student is invited to try himself to generalize to the d-dimensional case the definition of Knothe's map, its properties, and its use in the proof of the Isoperimetric inequality.