Serie 12

Optimal transport, Fall semester

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Exercise 12.1. Let $E \in C^1(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$ be such that the function $E(x) - \frac{\lambda}{2}||x||^2$ is convex. Show that a C^1 function u_t is a solution of $\partial_t u_t = -\nabla E(u_t)$ if and only if the evolutionary variational inequality (EVI)

$$\frac{1}{2}\frac{d}{dt}||u_t - v||^2 + \frac{\lambda}{2}||u_t - v||^2 + E(u_t) \leq E(v) \tag{1}$$

holds for any $v \in \mathbb{R}^d$.

Suppose now that we have two curves u_t and v_t satisfying $\partial_t u_t = -\nabla E(u_t)$. Prove that, if we define $d(t) = \|u_t - v_t\|^2$, then

$$d(t) \leq d(0)e^{-\lambda t}.$$

In particular, if $\lambda > 0$ and w_0 is the unique minimizer of E, then $\|u_t - w_0\|^2 \leq 2(\|u_0\|^2 + \|w_0\|^2)e^{-\lambda t}$.

Solution: Recall that $F : \mathbb{R}^d \to \mathbb{R}$ with $F \in C^1$ is convex if and only if for any $x, y \in \mathbb{R}^d$

$$F(y) \geq F(x) + \nabla F(x) \cdot (y - x).$$

In particular, a simple computation yields that $E(x) - \frac{\lambda}{2}\|x\|^2$ with $E \in C^1$ is convex if and only if for any $x, y \in \mathbb{R}^d$

$$E(y) \geq E(x) + \nabla E(x) \cdot (y - x) + \frac{\lambda}{2}\|y - x\|^2. \tag{2}$$

Suppose now that u_t solves $\partial_t u = -\nabla E(u)$. In particular, we have that for any $v \in \mathbb{R}^d$,

$$\frac{1}{2}\frac{d}{dt}||u_t - v||^2 = \langle \partial_t u_t, u_t - v\rangle = -\nabla E(u_t) \cdot (u_t - v).$$

That is, since $E(x) - \frac{\lambda}{2}\|x|\|^2$ is convex, plugging into (2) with $y = v$ and $x = u_t$ we get that(1) holds.

On the other hand, suppose that (1) holds for any $v \in \mathbb{R}^d$. Then we have (1), which from the previous computation is

$$E(v) \geq E(u_t) - \partial_t u_t \cdot (v - u_t) + \frac{\lambda}{2}\|v - u_t\|^2, \tag{3}$$

for all $v \in \mathbb{R}^d$. In particular, let $v = u_t + \tau\xi$ for $\tau > 0$ and $\xi \in \mathbb{S}^{d-1}$ fixed. Rewriting the previous expression we have

$$\frac{E(u_t + \tau\xi) - E(u_t)}{\tau} \geq -\partial_t u_t \cdot \xi + \frac{\lambda}{2}\tau.$$

We now let $\tau \downarrow 0$ to obtain

$$\nabla E(u_t) \cdot \xi \geq -\partial_t u_t \cdot \xi$$

since $E \in C^1$. Since we can take $-\xi$ instead of ξ, we get $\nabla E(u_t) \cdot \xi \geq -\partial_t u_t \cdot \xi$ for all $\xi \in \mathbb{S}^{d-1}$, that is, $\nabla E(u_t) = -\partial_t u_t$, as we wanted to see.

For the second part, we have that for any $t > 0$, from (3) applied to both u_t and v_t,

$$E(v_t) \geq E(u_t) - \partial_t u_t \cdot (v_t - u_t) + \frac{\lambda}{2}\|v_t - u_t\|^2,$$

and

$$E(u_t) \geq E(v_t) - \partial_t v_t \cdot (u_t - v_t) + \frac{\lambda}{2}\|u_t - v_t\|^2.$$

Adding both equations we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_t - v_t\|^2 = (\partial_t u_t - \partial_t v_t)(u_t - v_t) \leq -\frac{\lambda}{2}\|u_t - v_t\|^2.$$

By Grönwall's lemma, we are done.

In particular, solutions to $\partial_t u_t = -\nabla E(u_t)$ are unique once given the initial value u_0.

Suppose now that w_t is a curve such that w_0 is the unique minimizer of E. Then $\nabla E(w_0) \equiv 0$ so the constant curve $w_t = w_0$ is a solution (which is unique by the previous discussion). We immediately obtain the exponential convergence to the unique minimizer.

Exercise 12.2. Recall the Benamou-Brenier formula: given two probability measures $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$, then it holds that

$$W_2^2(\mu_0, \mu_1) = \inf\left\{\int_0^1 \int_{\mathbb{R}^d} |v_t|^2 d\rho_t\, dt : \partial_t \rho_t + \operatorname{div}(v_t \rho_t) = 0, \rho_0 = \mu_0, \rho_1 = \mu_1\right\}.$$

Suppose that μ_t for $t \in [0, 1]$ is a curve attaining the minimum, and suppose that $\mu_t = (X_t)_{\#}\mu_0$, for some smooth vector field X_t. Prove that $\ddot{X}_t \equiv 0$ μ_0-a.e. for a.e. $t \in (0, 1)$.

Solution: Let us start by observing that, if $\rho_t = (Y_t)_{\#}\mu_0$ with

$$\begin{cases} \dot{Y}_t &= w_t \circ Y_t \\ Y_0 &= \text{id}, \end{cases}$$

that is, $\partial_t \rho_t + \operatorname{div}(w_t \rho_t) = 0$, then we have

$$\int_{\mathbb{R}^d} |w_t|^2 \rho_t = \int_{\mathbb{R}^d} |w_t \circ Y_t|^2 \mu_0 = \int_{\mathbb{R}^d} \left|\dot{Y}_t\right|^2 \mu_0.$$

Let us consider the vector field $Y_{t,\varepsilon} := X_t + \varepsilon Z_t$, where Z_t is a smooth vector field, compactly supported in time. In particular, $Z_0 = Z_1 = 0$. Let us consider $\rho_{t,\varepsilon} := (Y_{t,\varepsilon})_{\#}\mu_0$, so that $\rho_{0,\varepsilon} = \mu_0$ and $\rho_{1,\varepsilon} = \mu_1$.

Then, by minimality, we have that

$$\int_0^1 \int_{\mathbb{R}^d} \left|\dot{X}_t\right|^2 d\mu_0\, dt \leq \int_0^1 \int_{\mathbb{R}^d} \left|\dot{Y}_{t,\varepsilon}\right|^2 d\mu_0\, dt = \int_0^1 \int_{\mathbb{R}^d} \left|\dot{X}_t\right|^2 d\mu_0\, dt + 2\varepsilon \int_0^1 \int_{\mathbb{R}^d} \dot{X}_t \cdot \dot{Z}_t d\mu_0\, dt + o(\varepsilon).$$

In particular, by taking $-Z_t$ instead of Z_t we deduce, letting $\varepsilon \downarrow 0$,

$$\int_0^1 \int_{\mathbb{R}^d} \dot{X}_t \cdot \dot{Z}_t d\mu_0 \, dt = 0.$$

By Fubini, integrating by parts in time first, and using that $Z_0 = Z_1 = 0$,

$$0 = \int_0^1 \int_{\mathbb{R}^d} \dot{X}_t \cdot \dot{Z}_t d\mu_0 \, dt = \int_{\mathbb{R}^d} \int_0^1 \dot{X}_t \cdot \dot{Z}_t dt \, d\mu_0 = - \int_{\mathbb{R}^d} \int_0^1 \ddot{X}_t \cdot Z_t \, dt \, d\mu_0.$$

From the arbitrariness of Z_t, we deduce the desired result.

Exercise 12.3. Let $\mu_0 = \rho_0 \mathscr{L}^d, \mu_1 = \rho_1 \mathscr{L}^d \in \mathcal{P}(\mathbb{T}^d)$ be two probability measures on the d-dimensional torus such that $\rho_0, \rho_1 \geq c > 0$ everywhere. Let $u : \mathbb{T}^d \to \mathbb{R}$ be a solution of the Poisson equation $-\Delta u = \rho_1 - \rho_0$. Show that

$$W_2(\mu_0, \mu_1) \leq c^{-1/2} \|\nabla u\|_{L^2}.$$

Hint: Use the Benamou-Brenier formula, which is valid also on the torus.

Solution: See solution in Figalli-Glaudo (last chapter).

Exercise 12.4. Let $U : [0, \infty) \to \mathbb{R}$ be a convex function with $U(0) = 0$ such that the energy functional $\mathscr{F}(\rho) := \int_{\mathbb{R}^d} U(\rho) dx$ is W_2-convex. Prove that the function $(0, \infty) \ni s \mapsto U(1/s^d) s^d$ is non-increasing and convex.

Solution: See solution in Figalli-Glaudo (last chapter).