Serie 7

Optimal transport, Fall semester

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Exercise 7.1 (Counterexamples)**.** For any of the following statements, find two probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ with compact support such that the statement holds (you can choose also the dimension $d \in \mathbb{N}$). Each of the statements should be treated independently.

(i) There is more than one[1] optimal transport map from μ to ν with respect to the linear cost $|x-y|$.

(ii) There is more than one optimal transport map from μ to ν with respect to the quadratic cost $\frac{1}{2}|x-y|^2$.

(iii) There is not an optimal transport plan between μ and ν with respect to the cost $c(x,y) = \lfloor |x-y| \rfloor$ (the floor function[2] of the distance).

(iv) There is an optimal transport map from μ to ν with respect to the linear cost, but there is none with respect to the quadratic cost.

Hint: To solve (iii), show that the infimum of the Kantorovich problem for $\mu = \chi_{[0,1]}\mathcal{L}^1,\ \nu = \chi_{[1,2]}\mathcal{L}^1$ is 0 but any transport plan has *strictly* positive cost.

Exercise 7.2. Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ be two compactly supported probability measures invariant under rotations (that is, $\mu(L(E)) = \mu(E)$ and $\nu(L(E)) = \nu(E)$ for any Borel set E and any orthogonal transformation $L \in O(d)$). Assume that $\mu \ll \mathcal{L}^d$, and let T be the unique optimal transport map from μ to ν with respect to the quadratic cost (see Theorem 2.5.9). Show that T can be written as $x \to \tau(|x|)\frac{x}{|x|}$, where $\tau : [0,+\infty) \to [0,+\infty)$ is a nondecreasing function.

Hint: The function τ is the monotone transport map between two suitable 1-dimensional measures. Also, one may want to use (and prove) the following lemma:

Lemma 1. Let $\mu^0, \mu^1 \in \mathcal{P}(\mathbb{R}^d)$ be two rotationally invariant probability measures, and let $\Phi(x) := |x|$. If $\Phi_\# \mu^0 = \Phi_\# \mu^1$ then $\mu^0 = \mu^1$.

Exercise 7.3. Find the optimal transport map for the quadratic cost $c(x,y) = \frac{1}{2}|x-y|^2$ between $\mu = f \cdot \mathcal{L}^2$ and $\nu = g \cdot \mathcal{L}^2$ in $\mathbb{R}^2$, where $f(x) = \frac{1}{\pi}\mathbb{1}_{B_1}(x)$ and $g(x) = \frac{1}{8\pi}(4-|x|^2)\mathbb{1}_{B_2}(x)$.

[1] Uniqueness should be understood in the μ-almost everywhere sense.

[2] Given $t \geq 0$, $\lfloor t \rfloor$ is the largest integer n such that $n \leq t$.

Remark 7.1. We gave in class a counterexample for the following statement: let us fix a cost c lower semicontinuous and $\gamma \in \Gamma(\mu, \nu)$ such that suppγ is c-cyclically monotone, then γ is optimal.

The example is the following: let $\mu = \mathcal{L}$ be the lebesgue measure on the one dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $c : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ be defined as follows

$$c(x, y) = \begin{cases} 1 & \text{if and only if } y = x - \alpha \\ 0 & \text{if and only if } y = x \\ \infty & \text{otherwise ,} \end{cases}$$

$\gamma = (Id, T)_{\#}\mu$ and $T(x) = x - \alpha$. We proved in class that suppγ is c-cyclically monotone but γ is not optimal.

Exercise 7.4. (💣) In this exercise we want to go deeper in the understanding of the counterexample in Remark 7.1.

(i) Find as many points as you can in the proof of the statement "suppγ is c-cyclically monotone, then γ is optimal for a continuous cost" where it could fail.

(ii) If we define for any $n \in \mathbb{N}$ the cost

$$c_n(x, y) = \begin{cases} 1 & \text{if and only if } y = x - \alpha \\ 0 & \text{if and only if } y = x \\ n & \text{otherwise,} \end{cases}$$

Is it true that γ is optimal for the cost c_n for any $n \in \mathbb{N}$ (where γ was defined above $\gamma = (Id, T)_{\#}\mu$)?

Find all the $n \in \mathbb{N}$ such that suppγ is c_n-cyclically monotone for the cost c_n?