Serie 5
Optimal transport, Fall semester
EPFL, Mathematics section, Dr. Xavier Fernández- Real

Exercise 5.1. Let $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$, $x_1 \neq x_2$ and let

$$\mu = \frac{1}{2}\delta_{x_1} + \frac{1}{2}\delta_{x_2} \quad \text{and} \quad \nu = \frac{1}{2}\delta_{y_1} + \frac{1}{2}\delta_{y_2}.$$

(i) Describe all maps transporting μ to ν; that is, such that $T_{\#}\mu = \nu$.

(ii) Describe all couplings of μ and ν; that is $\gamma \in \mathcal{P}(X \times Y)$ such that $(\pi_X)_{\#}\gamma = \mu$ and $(\pi_Y)_{\#}\gamma = \nu$.

(iii) Prove that, for any choice of continuous cost $c \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, there exists an optimal transport map (i.e., the optimal coupling has a map structure).

(iv) Assuming that x_1, Tx_1, x_2 and Tx_2 are not colinear, observe that for the linear cost $c(x, y) = |x - y|$, the corresponding optimal transport map does not cross trajectories.

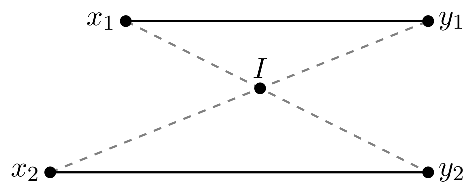

Figure 1: Crossing trajectories

Exercise 5.2. Say if the following sentences are true or false. If they are true, prove it, if they are false, provide a counterexample. The statements below all refer to the quadratic cost.

(i) Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a convex function, then φ is differentiable $\mathcal{L}^d$-a.e. in $\mathbb{R}^d$ and we call $N \subset \mathbb{R}^d$ the Lebesgue measure zero set where φ is not differentiable. For any $x \in N$ take an element $y_x \in \partial\varphi(x)$, and define the map $T : \mathbb{R}^d \to \mathbb{R}^d$ as follows:

$$T(x) = \begin{cases} \nabla\varphi(x) & \text{if } x \in \mathbb{R}^d \setminus N, \\ y_x & \text{if } x \in N. \end{cases}$$

Then, given $\mu \ll \mathcal{L}^d$, the map T is optimal from μ to $T_{\#}\mu$.

(ii) If $T : \mathbb{R} \to \mathbb{R}$ is an optimal map between μ_1 and μ_2 (i.e. $T_{\#}\mu_1 = \mu_2$) and $S : \mathbb{R} \to \mathbb{R}$ is optimal between μ_2 and μ_3, then $S \circ T$ is optimal between μ_1 and μ_3.

(iii) The same as before but in general dimension $d \geq 2$, namely: if $T : \mathbb{R}^d \to \mathbb{R}^d$ is an optimal map between μ_1 and μ_2 (i.e. $T_\# \mu_1 = \mu_2$) and $S : \mathbb{R}^d \to \mathbb{R}^d$ is optimal between μ_2 and μ_3, then $S \circ T$ is optimal between μ_1 and μ_3.

Exercise 5.3 (Birkhoff - Von Neumann Theorem)**.** A $(n \times n)$-matrix $A \in \mathscr{M}(n, \mathbb{R})$ with nonnegative entries is said to be:

- *a doubly-stochastic matrix* if $\sum_{i=1}^n A_{ij} = 1$ for any $j = 1, \ldots, n$, and $\sum_{j=1}^n A_{ij} = 1$ for any $i = 1, \ldots, n$.
- *a permutation matrix* if there is a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $A_{i\sigma(i)} = 1$ and $A_{ij} = 0$ if $j \neq \sigma(i)$.

Prove that any doubly-stochastic matrix can be written as a finite convex combination of permutation matrices.

Hints: Here is a guideline through a possible proof of the result:

- Use Hall's marriage Theorem[1] to prove that given a doubly-stochastic matrix A, there exists a permutation $\sigma \in S_n$ such that $A_{i\sigma(i)} > 0$ for any $i = 1, \ldots, n$. Deduce that there exists a permutation matrix P and $\lambda > 0$ such that $A_{ij} \geq \lambda P_{ij}$, $\forall i, j \in \{1, \ldots, n\}$.
- Let us now prove the result by induction on the number of non-zero entries k of A. Start by proving that $k \geq n$ and that the result holds for $k = n$.
- Let now $k > n$. Consider the permutation P and λ given in the first bullet above, and define

$$A' = \frac{1}{1 - \lambda}(A - \lambda P).$$

  Show that A' is doubly-stochastic with at most $k - 1$ non-zero entries.
- Deduce, by induction, that A is a convex combination of permutation matrices.

Exercise 5.4 (Discrete optimal transport)**.** Given two families $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ of points in $\mathbb{R}^d$, let $\mu := \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$ and $\nu := \frac{1}{n} \sum_{i=1}^n \delta_{y_i}$. Prove that, for *any* choice of a continuous cost $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, there exists an optimal transport map from μ to ν.

Hint: Use Exercise 5.3 or Kantorovich duality.

[1] See `https://en.wikipedia.org/wiki/Hall%27s_marriage_theorem#Graph_theoretic_formulation`.