Exam

- Timetable will be sent by e-mail (including room no / INM 203)
- 20-25 minutes oral
- Work on paper (not on blackboard)

Overview

Startingpoint:
- Underlying assets
- Stochastic processes

Definitions:
- Call / Put; European / American
- Forwards (Futures)
- Barrier options
- Contingent claim: $\mathcal{F}_N$, $\mathcal{F}_T$-measurable (suitably integrable) non-negative random variable.

Main tasks discussed in this course:
- pricing
- hedging

Focus on the theoretical basis

Key: Translation of the economical property of

"absence of arbitrage opportunities" $\longleftrightarrow$ existence of an "equivalent martingale measure"

$\Downarrow$

pricing rules

"arbitragefree" markets that are complete $\longleftrightarrow$ uniqueness of the "equivalent martingale measure"

Discrete time finite probability space

- d - risky assets
- 1 - riskless bond
- Strategies: predictable processes (sequences; "$\mathcal{F}_{n-1}$-measurable")
- Value process

$$V_n(\phi) = \sum_{i=0}^{d} \phi_n^i S_n^i = \phi_n \cdot S_n$$

or (to - arbitrage theory on finite probab. spaces) equivalently

$$\tilde{V}_n(\phi) = \frac{1}{S_n^0} \phi_n \cdot S_n = \phi_n \cdot \tilde{S}_n$$

- Self-financing: · Equivalent definitions (only initial costs, no investments and no consumption before maturity).
  - Determined by $(\phi_n^1, \dots, \phi_n^d)$ and V_0.
- Admissible strategies: Value of the portfolio must remain non-negative at all times and the strategy is self-financing.
- Arbitrage: Admissible strategy with $V_0 = 0$ and $P(V_N > 0) > 0$.
- FTAP I: viable (free of arbitrage) $\Longleftrightarrow$ $\exists Q$ equivalent to P s.t. $\tilde{S}$ is a Q-mart.
  (all components are Q martingales)

Tools: · Results from discrete time martingale th. (all proved)
· Convex separation th. / Hahn-Banach-Lin-Alg.-Version (not proved)

Proof: "$\Longleftarrow$" relatively easy (play with martingale properties and the admissibility property
"$\Longrightarrow$" relatively difficult (construction of an ex. measure)

- Complete markets: Every contingent claim is attainable

$\Updownarrow$

$\exists$ an admissible strategy worth H at N

$\uparrow$

$\mathcal{F}_N$-measurable, non-negative

- FTAP II: $\boxed{\text{A viable market}}$ is complete $\iff$ $\exists$ $\underset{=}{1}$ equivalent martingale measure

Proof: "$\Rightarrow$" relatively easy $(Q_1 = Q_2)$

"$\Leftarrow$" relatively difficult

(viable and incomplete $\Rightarrow$ we can construct another different equiv. m'measure. Hence, unique $\Rightarrow$ complete)

- Note: For pricing Q and not P is relevant.

- Pricing and hedging contingent claims in complete markets: No-arbitrage arguments can be applied very directly (as in B&S).

H - claim

$$V_n = S_n^0 E_Q\left(\frac{H}{S_N^0} \middle| \mathcal{F}_n\right)$$

Value of replicating strategy

- Based on the First Fundamental Theorem of Asset Pricing, this so-called "risk-neutral valuation" provides an arbitrage-free pricing rule for all equivalent martingale measures in an arbitrage-free and incomplete market (tower property).
- When choosing the specific martingale measure, practitioners often relay on calibration to market data.
- In arbitrage-free, incomplete markets, all martingale measures deliver the same value for $\boxed{\text{attainable}}$ claims.
- Slides 5: Analogous results in the "most general setting" (in the "classical understanding").
- Focus will be on Slides 1 and not on Slides 5.
- We have many exercises related to discrete time models.
- Before some easy general results, e.g. Forward pricing, European put-call parity, American options.

- CRR-model (binomial model) - complete
- Trinomial model as an example for incomplete markets

American options

- Short seller point of view ⇝ backward induction

$$U_{n-1} = \max\left(Z_{n-1},\ S^0_{n-1} E_Q\left(\frac{U_n}{S^0_n} \Big| \mathcal{F}_n\right)\right) \qquad ⚠ \neq Z_n$$

to pay if exercised

value to be replicated if the long position does not exercise (completeness is assumed)

$(\tilde{U}_n)$ - Snell envelope

- $(\tilde{U}_n)$ is the smallest Q supermartingale that dominates $(\tilde{Z}_n)$.
- How does that fit into our theory about pricing by exclusion of arbitrage?

Answer: By optimal stopping.

$\nu = \inf\{n \geq 0 : \tilde{U}_n = \tilde{Z}_n\}$ - stopping time

where $(\tilde{U}_{n \wedge \nu})$ - martingale

$U_0 = E_Q(\tilde{Z}_\nu) = \sup_{\tau \in \mathcal{T}_{0,N}} E_Q(\tilde{Z}_\tau)$ - hence, is optimal

It is also the smallest optimal stopping time

- $(\tilde{U}_n) = (\tilde{M}_n) - (\tilde{A}_n)$

Doob (unique) mart. non-decreasing predictable, $A_0 = 0$

largest optimal stopping time

$$\nu_{max} = \begin{cases} N & \text{if } \tilde{A}_N = 0 \\ \inf\{n : \tilde{A}_{n+1} \neq 0\} & \text{if } \tilde{A}_N \neq 0 \end{cases}$$

- Furthermore, $\left.\begin{array}{l} \cdot\ (\tilde{U}_n^{\nu}) \text{ - martingale} \\ \cdot\ \tilde{U}_\nu = \tilde{Z}_\nu \end{array}\right\}$ - is a characterization of the optimal stopping rules

Hence, if the long position exercises optimally, everything suits fine into the theory about absence of arbitrage.

More applied / exercises, e.g.

- Pricing on a binomial tree by backward induction
- Early ex. call with $r \geq 0$ and no-dividends (makes no sense)
- Early ex. of puts: Why this could make sense

But also: Doob decomposition.

B&S very rough

- Model

$$dS_t = \mu S_t\,dt + \sigma S_t\,dB_t$$ - SDE (B_t) sBM under P

$$dS_t^0 = r S_t^0\,dt$$ - determ. savings account

For given S_0, $S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right)$, (S_t) is the unique (strong) solution of the SDE.

- Strategies in this setting are adapted ($\mathbb{R}^2$-valued)

· Self-financing: Again analogue definitions, the meaning remains the same, e.g.

$$dV_t(\phi) = H_t^0\, dS_t^0 + H_t\, dS_t$$

$$\left(\int_0^T |H_t^0|\, dt < \infty \text{ a.s.} ;\ \int_0^T H_t^2\, dt < \infty \text{ a.s.} \right)$$

· In this setting: Admissible strategies are (adapted) self-financing with $\tilde{V}_t(\phi)$ being non-negative for all t and such that $E_Q\left(\int_0^T H_t^2 (\sigma S_t)^2\, dt \right) < \infty$, $\phi = (H_t^0, H_t)_{t \in [0,T]}$.

· If h is $\mathcal{F}_T$-measurable, non-negative and square-integrable under Q

$$\Downarrow$$

$$\boxed{V_t = e^{-r(T-t)} E_Q(h / \mathcal{F}_t) \qquad (1)}$$

Strategy to show that (use that the market is complete):

- Girsanov $\Rightarrow$ ex. Q being an equivalent martingale measure
- Martingale representation th.
  $\Rightarrow$ existence of a self-financing replication strategy
- no-arbitrage $\Longrightarrow$ (1) follows.

Hence, we can price claims.

- Example: · B&S for European calls
  - Barrier option (S.7, Ex. 4)
  - Sometimes there are possible shortcuts based on known results, e.g. European put via European put-call parity, or S.8, Ex. 3.

- Also the hedge can concretely be derived:

$$\begin{cases} \cdot H_t = \dfrac{\partial F}{\partial x}(t, S_t) \\ \cdot H_t^0 = e^{-rT} E_Q(h \mid \mathcal{F}_t) - H_t \tilde{S}_t \end{cases},$$

where $F(t, S_t) = e^{-r(T-t)} E_Q(h \mid \mathcal{F}_t)$ is our pricing formula ($\frac{\partial F}{\partial x}(t, S_t)$ is the delta)

- Hence, closed form formulas have some advantages for deriving the hedges. They are also useful for calibration

Slides 4

greeks and "model and parameter risk"
Implied vola

- 5.7: Reflection principle: Important applications, e.g. barrier options, model extensions (NG)
  not part of the exam
- There are many applied exercises for Ch. 3
- Slides 5: Idea discussed

Model Extensions

Explanations how to construct concrete models (Slides 5 deal with a general abstract theory)

- $\sigma \longrightarrow \sigma_t$ - Merton (deterministic σ.)
- $\sigma \longrightarrow \sigma(t, S_t)$ Dupire (deterministic $\sigma(\cdot,\cdot)$)

(complete markets / integral transforms)

There are also more complex integral transforms, in particular stochastic volatility. Integral transforms:

$$S_t = S_0 \exp\left(\int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dB_s\right)$$

or time changes

$$S_t = S_0 \exp\left(\mu T_t + B_{T_t}\right)$$

time change

Lévy-Processes

- Definition
- Why practical in valuation
- Roughly: Why are independent time changes often analytically particularly useful.
- Very rough idea of the construction of Generalized Hyperbolic Lévy processes

$$H_t = \mu t + \beta T_t + B_{T_t}$$

independent

$T_1 \sim$ Generalized Inverse Gaussian