Extra notes: Remarks on the theory of arbitrage for continuous-time models

Remarks on the vector stochastic integral:

Recall: $M = W$ - BM

$H = (H_t(\omega),\ t \geq 0,\ \omega \in \Omega\}$ - $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ - measurable and $(\mathcal{F}_t)$-adapted, with

$$E\left(\int_0^t H_s^2 \, ds\right) < \infty, \quad t \geq 0 \qquad (1)$$

Consider first simple functions

$$H_t(\omega) = \underbrace{h_0(\omega) \mathbb{1}_{\{0\}}(t) + \sum_{i \geq 0} h_i(\omega)}_{\mathcal{F}_{t_i}\text{-measurable and square integrable}} \mathbb{1}_{(t_i, t_{i+1}]}(t)$$

left continuous

There is a canonical definition of the stochastic integral given by

$$(H \cdot W)_t = \sum_{i \geq 0} h_i(\omega) \mathbb{1}_{(t_i, t_{i+1}]}(t) (W_{t_{i+1}} - W_{t_i})$$

②

One succeeds to construct simple functions H^n, $n \geq 0$ s.t.

$$E\left[\int_0^t (H_s - H_s^n)^2 ds\right] \xrightarrow{n\to\infty} 0, \quad t>0 \text{ fixed}$$

and therefore

$$E\left[\int_0^t (H_s^n - H_s^m)^2 ds\right] \xrightarrow[n,m\to\infty]{} 0, \quad t>0$$

A little calculation yields the Itô-Isometry for simple processes, i.e.

$$E\left[\left(\int_0^t H_s^n dW_s\right)^2\right] = E\left[\int_0^t (H_s^n)^2 ds\right], \quad t>0$$

Consequently,

$$E\left[\left(\int_0^t H_s^n dW_s - \int_0^t H_s^m dW_s\right)^2\right]$$

$$\|$$

$$E\left[\int_0^t (H_s^n - H_s^m)^2 ds\right] \xrightarrow{m,n\to\infty} 0, \quad t>0$$

Hence, $\left(\int_0^t H_s^n \, dW_s\right)_{n \geq 0}$, $t \geq 0$

is a Cauchy sequence in the space L^2 (square integrable random variables)

Since L^2 is complete we can define

fixed

$$\int_0^t H_s \, dW_s := L^2\text{-}\lim_{n \to \infty} \int_0^t H_s^n \, dW_s$$

often also denoted by $(H \cdot W)_t$ for the stochastic integral on $[0,t]$.

Then, ... extension to processes and localization so that we have an integral (broader version) for H-measurable and adapted with

$$P\left(\int_0^t H_s^2\,ds < \infty\right) = 1, \quad t > 0.$$

In essence, the same construction works if instead of a Wiener process W one takes a square integrable martingale M.

Replace (1) by

$$E\left(\int_0^t H_s^2\,d\langle M\rangle_s\right) < \infty, \; t > 0 \quad (\tilde{1})$$

By Doob: For a square integrable martingale there exists a unique increasing predictable process (starting at 0) $\langle M\rangle$ such that $M^2 - \langle M\rangle \in M_{loc}$

Natural question:

For which functions is there a sequence of simple functions H^n, $n \geq 0$

s.t.

$$E\left[\int_0^t (H_s - H_s^n)^2 \, d\langle M\rangle_s\right] \xrightarrow{n\to\infty} 0, \; t>0$$

Answer depends on properties of $\langle M\rangle$, where it is always OK for H - predictable s.t. $(\tilde{\Lambda})$ holds.

Remark:
Recall that all processes we consider are at least measurable and adapted.

- $\langle M\rangle$ - absolutely continuous w.r.t. the Lebesgue-measure
  ↳ (measurable and) adapted is OK

- $\langle M\rangle$ - P a.s. continuous
  ↳ measurable and $H(\tau(\omega), \omega)$ - $\mathcal{F}_\tau$ - measurable for all finite stopping times τ - OK, i.e in particular progressively measurable is OK.

- $\langle M\rangle$ - no extra regularity
  ↳ predictable OK

Analogously to the BM case use for simple functions

$$E\left[\left(\int_0^t H_s\,dM_s\right)^2\right] = E\left[\left(\int_0^t H_s^2\,d\langle M\rangle_s\right)\right]$$

$t > 0$

⋮

analogous arguments

Let $X = (X_1, \dots, X_d)$ be a d-dimensional semimartingale admitting a decomposition

$$X = X_0 + \underbrace{A}_{\substack{BV \\ A \in \mathcal{V}^d}} + \underbrace{M}_{\substack{\text{local martingale} \\ M \in \mathcal{M}^d_{loc}}}$$

1) Integral w.r.t. a local martingale

There exists a non-decreasing adapted process $C = (C_t)_{t \geq 0}$, $C_0 = 0$ and optional processes $c^{ij} = (c^{ij}_t)_{t \geq 0}$

measurable w.r.t. the optional σ-field (on $\Omega \times \mathbb{R}_+$) generated by the càdlàg (i.e. right-continuous with left-hand limits) $(\mathcal{F}_t)$-adapted (measurable) processes.

with $c^{ij}_t(\omega) = c^{ji}_t(\omega)$ $i, j \in \{1, \dots, d\}$ with $(c^{ij}_t(\omega))$ positively definite

such that

$$\underbrace{[M^i, M^j]}_{\text{quadratic covariation}} = \int_0^\cdot c^{ij}_s \, dC_s$$

$$\underset{\uparrow \text{result}}{=} \sum_{s \leq \cdot} \Delta M^i_s \Delta M^j_s + \underbrace{[M^{ic}, M^{jc}]}_{\text{continuous part}}$$

cf. Itô $M^1 = \int_0^\cdot b^1_s \, dB_s$, $M^2 = \int_0^\cdot b^2_s \, dB_s \Rightarrow [M^1, M^2]_t = \int_0^t b^1_s b^2_s \, ds$

We say that $H \in L^1(M)$ if

$H = (H^1, \dots, H^d)$

is predictable with

$\left(\text{i.e. } \{(\omega,t) : H_t(\omega) \in B\} \in \mathcal{P} \;\; \forall B \in \mathcal{B}(\mathbb{R}^d) \right)$

$$E\left[\left(\int_0^\infty \left(\sum_{i,j=1}^d H_s^i c_s^{ij} H_s^j \right) dC_s \right)^{1/2} \right] < \infty \qquad (2)$$

A stochastically simple integrand is a process H of the form

$$H_t = h_0 \mathbb{1}_{\{t=0\}} + \sum_{k=1}^m h_k \mathbb{1}_{\{\tau_k < t \leq \tau_{k+1}\}}, \quad t \geq 0$$

h_k: d-dimensional, bounded $\mathcal{F}_{\tau_k}$-measurable random vector

$0 \leq \tau_1 \leq \dots \leq \tau_{m+1}$ - stopping times

It turnes out that

1) The function

$$\|H\|_{L^1(M)} = E\left[\left(\int_0^\infty \left(\sum_{i,j=1}^d H_s^i c_s^{ij} H_s^j\right) dC_s\right)^{1/2}\right]$$

is a norm on $L^1(M)$ - defined via (2)

2) The set of simple integrands that belong to $L^1(M)$ is dense in $L^1(M)$ (by a monotone class argument)

For a simple integrand we define the integral $H \cdot M$ as

$$(H \cdot M)_t = \sum_{i=1}^d \sum_{k=1}^m h_k^i \left(M^i_{t \wedge \tau_{k+1}} - M^i_{t \wedge \tau_k}\right), \quad t \geq 0$$

Note that $(H \cdot M)_t$ takes values in $\mathbb{R}$ and not in $\mathbb{R}^d$.

It turns out that

$$(*) \quad [H\cdot M] = \int_0^{\cdot} \Big(\sum_{i,j=1}^{d} H_s^i \, c_s^{ij} H_s^j \Big) \, dC_s$$

$$\Big(\text{cf. } \Big[\int_0^{\cdot} \varphi_s b_s \, dB_s\Big]_t = \int_0^t \varphi_s^2 b_s^2 \, ds \Big)$$

Let $H \in L^1(M)$

Then there exist a sequence (H^n) of simple integrands that converges to H in $L^1(M)$

Consider now

$$H^1 = \{ M \in \underset{\substack{\text{local} \\ \text{martingales}}}{M_{loc}} : E\big(\sup_{t\geq 0} |M_t| \big) < \infty \}$$

with $\|M\|_{H^1} = E\big(\sup_{t\geq 0} |M_t| \big)$

(Banach space, i.e. normed and complete)

Davis inequalities: There exists constants $0 < c < C$ such that for any $M \in H^1$ with $M_0 = 0$

$$c E\big([M]_\infty^{1/2} \big) \leq E\big(\sup_{t\geq 0} |M_t| \big) \leq C E\big([M]_\infty^{1/2} \big)$$

Equation (*) combined with Davis inequalities yield that $(H^n \cdot M)$ is a Cauchy sequence in $\mathcal{H}^1$:

- (H^n) - sequence of stoch. simple integrands that converges to H in $L^1(M)$, i.e.

$$\|H - H^n\|_{L^1(M)}$$
$$\|$$
$$E\left[\left(\int_0^\infty \left(\sum_{i,j=1}^d (H_s^i - H_s^{in}) c_s^{ij} (H_s^j - H_s^{jn})\right) dC_s\right)^{1/2}\right] \xrightarrow{n\to\infty} 0$$

- Consider now

$$\|H^n \cdot M - H^m \cdot M\|_{H^1}$$

$$\|$$

$$E\left(\sup_{t\geq 0} \left|(H^n \cdot M)_t - (H^m \cdot M)_t\right|\right)$$

$$\|$$

$$E\left(\sup_{t\geq 0} \left|((H^n - H^m)\cdot M)_t\right|\right)$$

$$\wedge\!\!| \text{ - Davis}$$

$$cE\left[\left(\int_0^\infty \left(\sum_{i,j=1}^d (H_s^{in} - H_s^{im}) C_s^{ij} (H_s^{jn} - H_s^{jm})\right) dG_s\right)^{1/2}\right]$$

$$\Big\downarrow \; n,m \to \infty$$

$$0$$

Hence, there exists a $\mathcal{H}^1$-limit for this sequence $(H^n \cdot M)$

Thus, for $H \in L^1(M)$ the vector stochastic integral $H \cdot M$ is defined as the $\mathcal{H}^1$-limit of $(H^n \cdot M)$.

Extension by localisation; we say that $H \in L^1_{loc}(M)$ if it is predictable and if

$$\left(\int_0^{\cdot} \sum_{i,j=1}^{d} H_s^i c_s^{ij} H_s^j \, dC_s \right)^{1/2} \in \mathcal{A}_{loc}$$

$\mathcal{A}_{loc}$ - process of locally integrable variation, i.e. for $A \in \mathcal{A}_{loc}$ there exists an increasing sequence of stopping times such that $\tau_n \longrightarrow \infty$ a.s. and for all $n \in \mathbb{N}$,

$$E\left(\operatorname{Var}(A)_{\tau_n}\right) < \infty$$

$\operatorname{Var}(A)_t(\omega)$: total variation of $s \rightsquigarrow A_s(\omega)$ on $[0,t]$

$$\Big(\sup_{0 \le t_0 < \dots < t_n \le t} \sum_{i=1}^{n} |A_{t_i} - A_{t_{i-1}}|\Big)$$

Let $H \in L^1_{loc}(M)$. Then there exists an increasing sequence $(\tau_n)_{n=1}^{\infty}$ of stopping times such that $\tau_n \longrightarrow \infty$ a.s. and

$$E\left[\left(\int_0^{\tau_n} \Big(\sum_{i,j=1}^{d} H_s^i c_s^{ij} H_s^j\Big) dC_s\right)^{1/2}\right] < \infty .$$

For each $n \in \mathbb{N}$

$$H\mathbb{1}_{[[0,\tau_n]]} \in L^1(M).$$

Furthermore, there exists a unique process $H \cdot M$ such that

$$(H \cdot M)^{\tau_n} = (H\mathbb{1}_{[[0,\tau_n]]}) \cdot M, \; n \in \mathbb{N} \quad (**)$$

where $H \cdot M$ does not depend on the choice of the localizing sequence $(\tau_n)_{n=1}^{\infty}$.

Definition: For $H \in L^1_{loc}(M)$ the process $H \cdot M$ that satisfies $(**)$ is called the vector stochastic integral of H w.r.t. M.

We have $H \cdot M \in \mathcal{M}_{loc}$ and

$$[H \cdot M] = \int_0^{\cdot} \Big(\sum_{i,j=1}^{d} H_s^i c_s^{ij} H_s^j \Big) dC_s$$

2 Integral with respect to a process of finite variation

For $A \in \mathcal{V}^d$ there exists an adapted, increasing process C and an optional process a^i, $i = 1, \dots, d$ such that

$$A^i = \int_0^{\cdot} a_s^i \, dC_s$$

We say that predictable $H = (H^1, \dots, H^d)$ is in $L_{var}(A)$ if for any $t \geq 0$

$$\int_0^t \Big| \sum_{i=1}^d H_s^i a_s^i \Big| \, dC_s < \infty \quad \text{a.s.}$$

($L_{var}(A)$ does not depend on the choice of (a^i) and C)

For $H \in L_{var}(A)$ the vector stochastic integral is defined by

$$H \cdot A = \int_0^{\cdot} \sum_{i=1}^d H_s^i a_s^i \, dC_s$$

$\left(H \cdot A \in \mathcal{V},\ Var(H \cdot A) = \int_0^{\cdot} \Big| \sum_{i=1}^d H_s^i a_s^i \Big| \, dC_s \right)$

Integral with respect to a semimartingale

We denote the just defined integral w.r.t. $M \in \mathcal{M}^d_{loc}$ by

$(M)\, H \cdot M$ and the integral w.r.t. $A \in \mathcal{V}^d$ (d-dim. BV-processes) by $(LS)\, H \cdot A$.

Definition: Let X be a d dimensional semimartingale. A process H is X-integrable if there exists a decomposition $X = A + M$ with $A \in \mathcal{V}^d$ and $M \in \mathcal{M}^d_{loc}$ such that $H \in L_{var}(A) \cap L^1_{loc}(M)$.
In this case, the vector stochastic integral is defined by

$$H \cdot X = (LS)\, H \cdot A + (M)\, H \cdot M.$$

The space of X-integrable processes is denoted by $L(X)$.

Remarks

- For $Z \in \mathcal{V}^d \cap \mathcal{M}^d_{loc}$ and

$$H \in L_{var}(Z) \cap L^1_{loc}(Z) \implies (LS)\,H\cdot Z = (M)\,H Z$$

- If $H \in L(X)$ then it may not belong to $L_{var}(A) \cap L^1_{loc}(M)$ for all the decompositions $X = A + M$ with $A \in \mathcal{V}^d$ and $M \in \mathcal{M}^d_{loc}$.

- However, if $X = A + M = A' + M'$ with $A, A' \in \mathcal{V}^d$, $M, M' \in \mathcal{M}^d_{loc}$ and $H \in L_{var}(A) \cap L^1_{loc}(M) \cap L_{var}(A') \cap L^1_{loc}(M')$ Then

$$(LS)\,H\cdot A + (M)\,H\cdot M = (LS)\,H\cdot A' + (M)\,H\cdot M'$$

so that the integral is well-defined.

- Linearity in X holds

$$H\cdot(\alpha_1 X_1 + \alpha_2 \cdot X_2) = \alpha_1 (H\cdot X_1) + \alpha_2 (H\cdot X_2)$$

Linearity in H as well

$$(\alpha_1 H_1 + \alpha_2 H_2)\cdot X = \alpha_1 (H_1 \cdot X) + \alpha_2 (H_2 \cdot X)$$

etc.

E.g. $\{H\cdot X : H \in L(X)\}$ is closed (in the Émery topology)

Application of FTAP I

$(\tilde{S}_{t1}, \ldots, \tilde{S}_{tn})$ - Q-martingale

$$P_t = S_t^0 E_Q\left(\frac{C}{S_T^0} \Big| \mathcal{F}_t\right), \quad t \in [0,T]$$

C - new claim

$\tilde{P}_t = E_Q\left(\frac{C}{S_T^0} \Big| \mathcal{F}_t\right)$, by the tower property we have $\forall\, t \geq s$

$$E_Q(\tilde{P}_t | \mathcal{F}_s) = E_Q\left(E_Q\left(\frac{C}{S_T^0} \Big| \mathcal{F}_t\right) \Big| \mathcal{F}_s\right)$$
$$= E_Q\left(\frac{C}{S_T^0} \Big| \mathcal{F}_s\right) = \tilde{P}_s$$
$$\Downarrow$$

$(\tilde{S}_{t1}, \ldots, \tilde{S}_{tn}, \tilde{P}_t)$ - Q-martingale

(note that Q is fixed in the first line)

Thus, for all equivalent σ-martingale measures

$S_t^0 E_Q\left(\frac{C}{S_T^0} \Big| \mathcal{F}_t\right)$ yields an arbitrage free pricing rule.